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NOTE ON CAUCHY'S NUMBERS.

By Prof. A. S. Chessin, Baltimore, Md.

Let p be any integer *positive, negative, or zero;* q and j any *positive integers or zero;* develop the expression

$$x^{-p}\left[x+\frac{1}{x}\right]^j\left[x-\frac{1}{x}\right]^q \tag{A}$$

in powers of x and denote the constant term of this development by $N_{-p,j,q}$. The number $N_{-p,j,q}$ is called a *number of Cauchy*. These numbers play an important role in certain developments used in Celestial Mechanics. In this note a formula will be given by which Cauchy's numbers can be directly and easily calculated.

First, let us recall some important properties of these numbers; namely, that

$$N_{-p,j,q}=1, \tag{1}$$

when $-p+j+q=0$;

$$N_{-p,j,q}=0, \tag{2}$$

when $-p+j+q$ is a negative number or when it is odd;

$$N_{p,j,q}=(-1)^q N_{-p,j,q}; \tag{3}$$

$$N_{-p,j+1,q}=N_{-p+1,j,q}+N_{-p-1,j,q}; \tag{4}$$

$$N_{-p,j,q+1}=N_{-p+1,j,q}-N_{-p-1,j,q}. \tag{5}$$

From the last two formulas, by a successive application of the same, the following are easily obtained:

$$N_{-p,j,q}=N_{-p+m,j-m,q}+\begin{bmatrix}m\\1\end{bmatrix}N_{-p+m-2,j-m,q}+\begin{bmatrix}m\\2\end{bmatrix}N_{-p+m-4,j-m,q}$$
$$+\begin{bmatrix}m\\3\end{bmatrix}N_{-p+m-6,j-m,q}+\ldots+\begin{bmatrix}m\\1\end{bmatrix}N_{-p-m+2,j-m,q}+N_{-p-m,j-m,q}, \tag{6}$$

$$N_{-p,j,q}=N_{-p+m,j,q-m}-\begin{bmatrix}m\\1\end{bmatrix}N_{-p+m-2,j,q-m}+\begin{bmatrix}m\\2\end{bmatrix}N_{-p+m-4,j,q-m}$$
$$-\begin{bmatrix}m\\3\end{bmatrix}N_{-p+m-6,j,q-m}+\ldots+(-1)^m N_{-p-m,j,q-m}. \tag{7}$$

In particular, if $m = j$, formula (6) gives

$$N_{-p,j,q} = N_{-p+j,0,q} + \begin{bmatrix} j \\ 1 \end{bmatrix} N_{-p+j-2,0,q} + \begin{bmatrix} j \\ 2 \end{bmatrix} N_{-p+j-4,0,q} + \begin{bmatrix} j \\ 3 \end{bmatrix} N_{-p+j-6,0,q} + \ldots + N_{-p-j,0,q}\,; \tag{6'}$$

and if $m = q$, formula (7) gives

$$N_{-p,j,q} = N_{-p+q,j,0} - \begin{bmatrix} q \\ 1 \end{bmatrix} N_{-p+q-2,j,0} + \begin{bmatrix} q \\ 2 \end{bmatrix} N_{-p+q-4,j,0} - \begin{bmatrix} q \\ 3 \end{bmatrix} N_{-p+q-6,j,0} + \ldots + (-1)^q N_{-p-q,j,0}\,. \tag{7'}$$

Combining formulas (6′) and (7′) together, we easily obtain the following:

$$\begin{aligned} N_{-p,j,q} = {} & N_{-p+j+q,0,0} + (j-q)\, N_{-p+j+q-2,0,0} \\ & + \left[\begin{bmatrix} j \\ 2 \end{bmatrix} - \begin{bmatrix} j \\ 1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} + \begin{bmatrix} q \\ 2 \end{bmatrix} \right] N_{-p+j+q-4,0,0} + \ldots \\ & + \left[\begin{bmatrix} j \\ n \end{bmatrix} - \begin{bmatrix} j \\ n-1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} + \begin{bmatrix} j \\ n-2 \end{bmatrix} \begin{bmatrix} q \\ 2 \end{bmatrix} - \ldots + (-1)^n \begin{bmatrix} q \\ n \end{bmatrix} \right] N_{-p+j+q-2n,0,0} \\ & + \ldots + (-1)^q N_{-p-j-q,0,0}\,. \end{aligned} \tag{8}$$

But

$$N_{-p+j+q-2n,0,0} = 0\,, \text{ if } -p+j+q-2n \gtrless 0\,,$$

$$N_{-p+j+q-2n,0,0} = 1\,, \text{ if } -p+j+q-2n = 0\,.$$

In fact, if $j = q = 0$ in formula (A), the constant part in the development is zero, unless $p = 0$, in which case it is unity. Hence

$$N_{-p,j,q} = \begin{bmatrix} j \\ n \end{bmatrix} - \begin{bmatrix} j \\ n-1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} + \begin{bmatrix} j \\ n-2 \end{bmatrix} \begin{bmatrix} q \\ 2 \end{bmatrix} - \ldots + (-1)^n \begin{bmatrix} q \\ n \end{bmatrix}$$

$$n = \tfrac{1}{2}(-p+j+q)\,.$$

This formula gives the solution of the proposed problem.

JOHNS HOPKINS UNIVERSITY, *October*, 1895.

ON THE FUNDAMENTAL PROPERTY OF THE LINEAR GROUP OF TRANSFORMATION IN THE PLANE.

By Dr. Arnold Emch, Lawrence, Kas.

A general projective transformation in the plane can easily be executed by means of two conics K and K' tangent to a certain straight line l in the following manner* (Fig. 1).

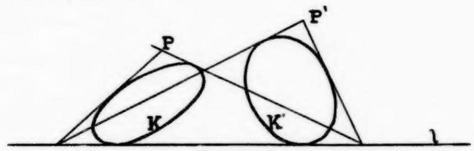

Figure 1.

From the point P to be transformed draw the two tangents to the conic K, and from their points of intersection with the line l draw the two possible tangents to the conic K'. The point of intersection of these two tangents is the point P' corresponding to P in the transformation. The invariant triangle of the transformation is obtained by the three other common tangents of the conics K and K'.

Now it is known that the linear transformation leaves the line at infinity invariant. Consequently, in a linear transformation, the conics K and K' must be parabolas.

By construction, or from the fact that every point of the line at infinity is transformed into another point of the line at infinity it follows that parallel lines are transformed into parallel lines. This property of the linear transformation is sufficient to prove in a simple way the well known theorem :

If in a projective transformation of the plane parallel lines are transformed into parallel lines, the areas of any two corresponding closed figures have a constant ratio.

To prove this we can consider two corresponding triangles Δ and Δ', however small (ABC and $A'B'C'$ in Fig. 2). Through each vertex of these triangles draw a line parallel to the opposite side and complete the net formed by parallels as is indicated in the figure.

* For this proposition we refer to an unpublished paper of Prof. Newson.

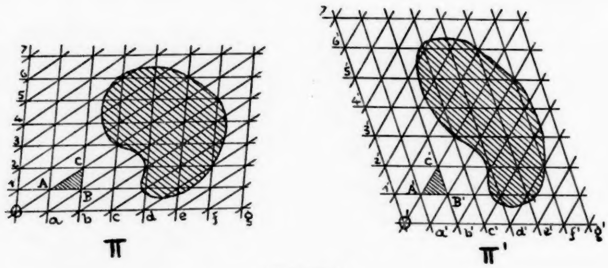

FIGURE 2.

Evidently, the points

$$O \text{ and } O'$$

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ \ldots$$

and

$$a',\ b',\ c',\ d',\ e',\ f',\ g',\ \ldots;$$

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$$

and

$$1',\ 2',\ 3',\ 4',\ 5',\ 6',\ 7',\ \ldots;$$

and the systems of parallel lines through these points are corresponding points and systems in the transformation.

Thus, the plane Π is divided into a net of equal triangles and the corresponding plane Π' into a net of corresponding equal triangles, such that any two corresponding triangles, or closed figures consisting of corresponding triangles, have the same constant ratio.

From this follows that any closed curve in the plane Π includes the same number of primitive triangles and parts of such triangles as the corresponding curve in the plane Π'. Designating the number of integral triangles within the curves by n, the sum of the fractional triangles within the curves by R and R', respectively, and the constant ratio by k, there is

$$\frac{n\triangle + R}{n\triangle' + R'} = k\,, \text{ or } \quad \frac{\triangle + \frac{R}{n}}{\triangle' + \frac{R'}{n}} = k\,.^*$$

Taking the limits, i. e., n infinitely large, or the triangles Δ and Δ' smaller than any finite quantity, this ratio becomes

$$\frac{\triangle}{\triangle'} = k\,, \text{ which was to be shown.}$$

* Compare Dirichlet's *Vorlesungen über Zahlentheorie*, 4th Ed., § 120, p. 310.

NOTE ON THE GENERAL PROJECTIVE TRANSFORMATION.

By Dr. E. O. Lovett, Leipzig, Germany.

The general projective transformation in space of three dimensions is an operation that carries every plane into the position of some plane. This second position of the plane may coincide with the first position; the transformation is then the identical transformation. Two planes with their line of intersection will be transferred to the position of two planes with a corresponding line of intersection; hence every right line is carried into the position of some right line, i. e. the family of ∞^4 right lines in space is invariant by the general projective transformation. In general no individual right line is invariant, but the family as a whole is invariant.

It is proposed to make use of this property, namely, the invariance of the right line by the transformation, to determine the finite forms of the general projective transformation in ordinary space.*

Let the equations of the right line be those of the two planes:

$$E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0\,,$$

$$E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0\,;$$

then without loss of generality these may be taken in their equivalent forms:

$$\begin{aligned} y &= ax + b\,, \\ z &= cx + d\,. \end{aligned} \tag{1}$$

By differentiation and elimination these equations give

$$\frac{d^2y}{dx^2} \equiv y'' = 0\,, \quad \frac{d^2z}{dx^2} \equiv z'' = 0\,. \tag{2}$$

Then the ∞^4 right lines in space represented by the equations (1) are the integral-curves of the simultaneous differential equations of the second order (2). Hence these two differential equations (2) are themselves invariant by the transformation.†

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* The general form of the projective transformation of the plane is derived from the invariance of the right line in the plane in Lie's "Vorlesungen über continuierliche Gruppen," pp. 33 et seq. —Leipzig, 1893.

† According to the theory of invariant differential equations a system of simultaneous differential equations is said to be invariant under a given transformation when the geometrical representatives of the given system constitute an invariant family. Thus the criterion that a given system be invariant is that the transformed equations shall represent the same family of geometrical figures that the original equations represented.

Now let the general projective transformation be expressed analytically by

$$x_1 = \varphi(x, y, z), \quad y_1 = \psi(x, y, z), \quad z_1 = \omega(x, y, z). \tag{3}$$

Then the transformation (3) transforms right line into right line, when φ, ψ and ω are such functions of x, y, z that the differential equations (2) have the same forms in the new variables, namely,

$$\frac{d^2y_1}{dx_1^2} = y_1'' = 0, \quad \frac{d^2z_1}{dx_1^2} \equiv z_1'' = 0, \tag{4}$$

as they had in the old variables x, y, z. This condition (4) furnishes a means of determining the forms of the functions φ, ψ, ω.

Equations (3) give by differentiation

$$\begin{aligned} \frac{dy_1}{dx_1} &= \frac{\psi_x + \psi_y y' + \psi_z z'}{\varphi_x + \varphi_y y' + \varphi_z z'}, \\ \frac{dz_1}{dx_1} &= \frac{\omega_x + \omega_y y' + \omega_z z'}{\varphi_x + \varphi_y y' + \varphi_z z'}. \end{aligned} \tag{5}$$

Further,

$$\begin{aligned} \frac{d^2y_1}{dx_1^2} = \frac{d\,\dfrac{dy_1}{dx_1}}{dx} : \frac{dx_1}{dx} = \frac{1}{(\varphi_x + \varphi_y y' + \varphi_z z')^3}\{&(\varphi_x + \varphi_y y' + \varphi_z z')(\psi_{xx} + 2\psi_{xy}y' \\ &+ \psi_{yy}y'^2 + \psi_y y'' + 2\psi_{yz}y'z' + 2\psi_{zx}z' + \psi_{zz}z'^2 + \psi_z z'') - (\psi_x + \psi_y y' \\ &+ \psi_z z')(\varphi_{xx} + 2\varphi_{xy}y' + \varphi_{yy}y'^2 + \varphi_y y'' + 2\varphi_{yz}y'z' + 2\varphi_{zx}z' + \varphi_{zz}z'^2 \\ &+ \varphi_z z'')\} \\ = N\{&(\varphi_x\psi_{xx} - \psi_x\varphi_{xx}) + (\varphi_y\psi_{xx} + 2\varphi_x\psi_{xy} - \psi_y\varphi_{xx} - 2\psi_x\varphi_{xy})y' \\ &+ (\varphi_z\psi_{xx} + 2\varphi_x\psi_{zx} - \psi_z\varphi_{xx} - 2\psi_x\varphi_{zx})z' \\ &+ (\varphi_x\psi_{yy} + 2\varphi_y\psi_{xy} - \psi_x\varphi_{yy} - 2\psi_y\varphi_{xy})y'^2 \\ &+ (\varphi_x\psi_{zz} + 2\varphi_z\psi_{zx} - \psi_x\varphi_{zz} - 2\psi_z\varphi_{zx})z'^2 \\ &+ 2(\varphi_x\psi_{yz} + \varphi_y\psi_{zx} + \varphi_z\psi_{xy} - \psi_x\varphi_{yz} - \psi_y\varphi_{zx} - \psi_z\varphi_{xy})y'z' \\ &+ (\varphi_y\psi_{yy} - \psi_y\varphi_{yy})y'^3 + (\varphi_z\psi_{zz} - \psi_z\varphi_{zz})z'^3 \\ &+ (\varphi_y\psi_{zz} + 2\varphi_z\psi_{yz} - \psi_y\varphi_{zz} - 2\psi_z\varphi_{yz})y'z'^2 \\ &+ (\varphi_z\psi_{yy} + 2\varphi_y\psi_{yz} - \psi_z\varphi_{yy} - 2\psi_y\varphi_{yz})z'y'^2\}, \end{aligned} \tag{6}$$

since $y'' = z'' = 0$, where $N = \dfrac{1}{(\varphi_x + \varphi_y y' + \varphi_z z')^3}$, $\varphi_x = \dfrac{\partial\varphi}{\partial x}$, $\varphi_{xx} = \dfrac{\partial^2\varphi}{\partial x^2}$, $y' = \dfrac{dy}{dx}$, etc.

Similarly

$$\begin{aligned}\frac{d^2z_1}{dx_1^2} = N\{&(\varphi_x\omega_{xx} - \omega_x\varphi_{xx}) + (\varphi_y\omega_{xx} + 2\varphi_x\omega_{xy} - \omega_y\varphi_{xx} - 2\omega_x\varphi_{xy})\,y' \\ &+ (\varphi_z\omega_{xx} + 2\varphi_x\omega_{zx} - \omega_z\varphi_{xx} - 2\omega_x\varphi_{zx})\,z' \\ &+ (\varphi_x\omega_{yy} + 2\varphi_y\omega_{xy} - \omega_x\varphi_{yy} - 2\omega_y\varphi_{xy})\,y'^2 \\ &+ (\varphi_x\omega_{zz} + 2\varphi_z\omega_{zx} - \omega_x\varphi_{zz} - 2\omega_z\varphi_{zx})\,z'^2 \\ &+ 2\,(\varphi_x\omega_{yz} + \varphi_y\omega_{zx} + \varphi_z\omega_{xy} - \omega_x\varphi_{yz} - \omega_y\varphi_{zx} - \omega_z\varphi_{xy})\,y'z' \\ &+ (\varphi_y\omega_{yy} - \omega_y\varphi_{yy})\,y'^3 + (\varphi_z\omega_{zz} - \omega_z\varphi_{zz})\,z'^3 \\ &+ (\varphi_y\omega_{zz} + 2\varphi_z\omega_{yz} - \omega_y\varphi_{zz} - 2\omega_z\varphi_{yz})\,y'z'^2 \\ &+ (\varphi_z\omega_{yy} + 2\varphi_y\omega_{yz} - \omega_z\varphi_{yy} - 2\omega_y\varphi_{yz})\,z'y'^2\}\,. \end{aligned} \tag{7}$$

These expressions (6) and (7) must be equal to zero for all values of x, y, z, y', z' by virtue of $y_1'' = 0$ and $z_1'' = 0$. Therefore, the conditions that y'' and z'' be invariant are given by the following system of partial differential equations:

$$\left.\begin{aligned} \varphi_x\psi_{xx} - \psi_x\varphi_{xx} &= 0\,, \\ \varphi_x\omega_{xx} - \omega_x\varphi_{xx} &= 0\,, \\ \varphi_y\psi_{yy} - \psi_y\varphi_{yy} &= 0\,, \\ \varphi_y\omega_{yy} - \omega_y\varphi_{yy} &= 0\,, \\ \varphi_z\psi_{zz} - \psi_z\varphi_{zz} &= 0\,, \\ \varphi_z\omega_{zz} - \omega_z\varphi_{zz} &= 0\,; \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} \varphi_y\psi_{xx} - \psi_y\varphi_{xx} + 2\,(\varphi_x\psi_{xy} - \psi_x\varphi_{xy}) &= 0\,, \\ \varphi_y\omega_{xx} - \omega_y\varphi_{xx} + 2\,(\varphi_x\omega_{xy} - \omega_x\varphi_{xy}) &= 0\,, \\ \varphi_z\psi_{xx} - \psi_z\varphi_{xx} + 2\,(\varphi_x\psi_{zx} - \psi_x\varphi_{zx}) &= 0\,, \\ \varphi_z\omega_{xx} - \omega_z\varphi_{xx} + 2\,(\varphi_x\omega_{zx} - \omega_x\varphi_{zx}) &= 0\,, \\ \varphi_x\psi_{yy} - \psi_x\varphi_{yy} + 2\,(\varphi_y\psi_{xy} - \psi_y\varphi_{xy}) &= 0\,, \\ \varphi_x\omega_{yy} - \omega_x\varphi_{yy} + 2\,(\varphi_y\omega_{xy} - \omega_y\varphi_{xy}) &= 0\,, \\ \varphi_x\psi_{zz} - \psi_x\varphi_{zz} + 2\,(\varphi_z\psi_{zx} - \psi_z\varphi_{zx}) &= 0\,, \\ \varphi_x\omega_{zz} - \omega_x\varphi_{zz} + 2\,(\varphi_z\omega_{zx} - \omega_z\varphi_{zx}) &= 0\,; \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned}
\varphi_y\psi_{zz} - \psi_y\varphi_{zz} + 2(\varphi_z\psi_{yz} - \psi_z\varphi_{yz}) &= 0\,,\\
\varphi_y\omega_{zz} - \omega_y\varphi_{zz} + 2(\varphi_z\omega_{yz} - \omega_z\varphi_{yz}) &= 0\,,\\
\varphi_z\psi_{yy} - \psi_z\varphi_{yy} + 2(\varphi_y\psi_{yz} - \psi_y\varphi_{yz}) &= 0\,,\\
\varphi_z\omega_{yy} - \omega_z\varphi_{yy} + 2(\varphi_y\omega_{yz} - \omega_y\varphi_{yz}) &= 0\,,\\
\varphi_x\psi_{yz} - \psi_x\varphi_{yz} + \varphi_y\psi_{zx} - \psi_y\varphi_{zx} + \varphi_z\psi_{xy} - \psi_z\varphi_{xy} &= 0\,,\\
\varphi_x\omega_{yz} - \omega_x\varphi_{yz} + \varphi_y\omega_{zx} - \omega_y\varphi_{zx} + \varphi_z\omega_{xy} - \omega_z\varphi_{xy} &= 0\,.
\end{aligned}\right\} \tag{10}$$

Hence, in order that the equations (4) may represent the general projective transformation in three variables it is necessary and sufficient that the functions φ, ψ and ω satisfy the system of partial differential equations (8), (9), (10). Then the integration of this system will give the forms of φ, ψ and ω.

From the equations (8) it follows that

$$\frac{\partial \log \psi_x}{\partial x} = \frac{\partial \log \varphi_x}{\partial x} = \frac{\partial \log \omega_x}{\partial x}\,,$$
$$\frac{\partial \log \psi_y}{\partial y} = \frac{\partial \log \varphi_y}{\partial y} = \frac{\partial \log \omega_y}{\partial y}\,,$$
$$\frac{\partial \log \psi_z}{\partial z} = \frac{\partial \log \varphi_z}{\partial z} = \frac{\partial \log \omega_z}{\partial z}\,.$$

Therefore,

$$\left.\begin{aligned}
\psi_x = P(y, z)\cdot\varphi_x\,,\quad \psi_y = Q(z, x)\cdot\varphi_y\,,\quad \psi_z = R(x, y)\cdot\varphi_z\,,\\
\omega_x = S(y, z)\cdot\varphi_x\,,\quad \omega_y = T(z, x)\cdot\varphi_y\,,\quad \omega_z = U(x, y)\cdot\varphi_z\,,
\end{aligned}\right. \tag{11}$$

in which P and S are functions of y and z alone; Q and T of z and x alone; R and U of x and y alone.

By substituting these values in equations (9) the latter become

$$\left.\begin{aligned}
(P - Q)\,\varphi_{xx}\varphi_y + 2P_y\varphi_x^2 &= 0\,,\\
(P - Q)\,\varphi_{yy}\varphi_x - 2Q_x\varphi_y^2 &= 0\,,\\
(P - R)\,\varphi_{xx}\varphi_z + 2P_z\varphi_x^2 &= 0\,,\\
(P - R)\,\varphi_{zz}\varphi_x - 2R_x\varphi_z^2 &= 0\,,\\
(S - T)\,\varphi_{xx}\varphi_y + 2S_y\varphi_x^2 &= 0\,,\\
(S - T)\,\varphi_{yy}\varphi_x - 2T_x\varphi_y^2 &= 0\,,\\
(S - U)\,\varphi_{xx}\varphi_z + 2U_z\varphi_x^2 &= 0\,,\\
(S - U)\,\varphi_{zz}\varphi_x - 2U_x\varphi_z^2 &= 0\,.
\end{aligned}\right. \tag{12}$$

Further, since $\psi_{xy} = \psi_{yx}$, $\psi_{xz} = \psi_{zx}$, $\omega_{xy} = \omega_{yx}, \ldots$, equations (11) give

$$
\begin{aligned}
(P - Q)\,\varphi_{xy} + P_y\varphi_x - Q_x\varphi_y &= 0\,, \\
(P - R)\,\varphi_{xz} + P_z\varphi_x - R_x\varphi_z &= 0\,, \\
(Q - R)\,\varphi_{yz} + Q_z\varphi_y - R_y\varphi_z &= 0\,, \\
(S - T)\,\varphi_{xy} + S_y\varphi_x - T_x\varphi_y &= 0\,, \\
(S - U)\,\varphi_{xz} + S_z\varphi_x - U_x\varphi_z &= 0\,, \\
(T - U)\,\varphi_{yz} + T_z\varphi_y - U_y\varphi_z &= 0\,.
\end{aligned}
\tag{13}
$$

By differentiation of the equations (12) partially with respect to x, y, x, z, x, y, x, z, respectively, we have the following relations:

$$Q_x\varphi_{xx}\varphi_y - (P - Q)\,(\varphi_{xxx}\varphi_y + \varphi_{xx}\varphi_{xy}) - 4P_y\varphi_x\varphi_{xx} = 0\,, \tag{14}$$

$$P_y\varphi_{yy}\varphi_x + (P - Q)\,(\varphi_{yyy}\varphi_x + \varphi_{yy}\varphi_{xy}) - 4Q_x\varphi_y\varphi_{yy} = 0\,, \tag{15}$$

$$R_x\varphi_{xx}\varphi_z - (P - R)\,(\varphi_{xxx}\varphi_z + \varphi_{xx}\varphi_{xz}) - 4P_z\varphi_x\varphi_{xx} = 0\,, \tag{16}$$

$$P_z\varphi_{zz}\varphi_x + (P - R)\,(\varphi_{zzz}\varphi_x + \varphi_{zz}\varphi_{xz}) - 4R_x\varphi_z\varphi_{zz} = 0\,; \tag{17}$$

together with four similar equations in which S, T, U, take the place of P, Q, R, respectively.

Equations (12) and (13) give the following:

$$\varphi_{xx} = \frac{-\,2P_y\varphi_x^2}{(P - Q)\,\varphi_y}\,, \tag{18}$$

$$\varphi_{xy} = \frac{Q_x\varphi_y - P_y\varphi_x}{P - Q}\,, \tag{19}$$

$$\varphi_{yy} = \frac{2Q_x\varphi_y^2}{(P - Q)\,\varphi_x}\,, \tag{20}$$

$$\varphi_{xz} = \frac{R_x\varphi_z - P_z\varphi_x}{P - R}\,, \tag{21}$$

$$\varphi_{zz} = \frac{2R_x\varphi_z^2}{(P - R)\,\varphi_x}\,. \tag{22}$$

Substituting (18) and (19) in (14):

$$
\begin{aligned}
&\frac{Q_x\varphi_y \,.\, 2P_y\varphi_x^2}{(P - Q)\,\varphi_y} + (P - Q)\,\varphi_{xxx}\varphi_y \\
&- \frac{2P_y\varphi_x^2(Q_x\varphi_y - P_y\varphi_x)}{(P - Q)\,\varphi_y} - \frac{8P_y\varphi_x P_y\varphi_x^2}{(P - Q)\,\varphi_y} = 0\,,
\end{aligned}
\tag{23}
$$

Substituting (19) and (20) in (15),

$$\frac{P_y\varphi_x \,.\, 2Q_x\varphi_y^2}{(P-Q)\varphi_x} + (P-Q)\varphi_{yyy}\varphi_x$$

$$+\frac{2Q_x\varphi_y^2(Q_x\varphi_y - P_y\varphi_x)}{(P-Q)\varphi_x} - \frac{8Q_x\varphi_y Q_x\varphi_y^2}{(P-Q)\varphi_x} = 0\,, \tag{24}$$

Substituting (21) and (22) in (17),

$$\frac{P_z\varphi_x \,.\, 2R_x\varphi_z^2}{(P-R)\varphi_x} + (P-R)\varphi_{zzz}\varphi_x$$

$$+\frac{2R_x\varphi_z^2(R_x\varphi_z - P_z\varphi_x)}{(P-R)\varphi_x} - \frac{8R_x\varphi_z \,.\, R_x\varphi_z^2}{(P-R)\varphi_x} = 0\,, \tag{25}$$

These equations become, after obvious reductions,

$$\varphi_{xxx} = \frac{6P_y^2}{(P-Q)^2}\frac{\varphi_x^3}{\varphi_y^2}, \tag{26}$$

$$\varphi_{yyy} = \frac{6Q_x^2}{(P-Q)^2}\frac{\varphi_y^3}{\varphi_x^2}, \tag{27}$$

$$\varphi_{zzz} = \frac{6R_x^2}{(P-R)^2}\cdot\frac{\varphi_z^3}{\varphi_x^2}. \tag{28}$$

Solving the first, second, and fourth equations of (12) for $P_y/(P-Q)$, $Q_x/(P-Q)$, $R_x/(P-R)$, respectively, and substituting in the last three equations, they become

$$\frac{\varphi_{xxx}}{\varphi_{xx}} = \frac{3}{2}\frac{\varphi_{xx}}{\varphi_x}, \quad \frac{\varphi_{yyy}}{\varphi_{yy}} = \frac{3}{2}\frac{\varphi_{yy}}{\varphi_y}, \quad \frac{\varphi_{zzz}}{\varphi_{zz}} = \frac{3}{2}\frac{\varphi_{zz}}{\varphi_z}. \tag{29}$$

Integrating these equations we have, $\frac{\varphi_{xx}^2}{\varphi_x^3}$ a function of y and z alone; $\frac{\varphi_{yy}^2}{\varphi_y^3}$ a function of x and z alone; $\frac{\varphi_{zz}^2}{\varphi_z^3}$ a function of x and y alone. The same is obviously true of $\frac{\varphi_{xx}}{\varphi_x^{\frac{3}{2}}}, \frac{\varphi_{yy}}{\varphi_y^{\frac{3}{2}}}, \frac{\varphi_{zz}}{\varphi_z^{\frac{3}{2}}}$; therefore the integrals of these $\frac{1}{\varphi_x^{\frac{1}{2}}}, \frac{1}{\varphi_y^{\frac{1}{2}}}, \frac{1}{\varphi_z^{\frac{1}{2}}}$ are linear in x, y, and z respectively; and therefore φ_x, φ_y, φ_z have the forms,

$$\varphi_x = \frac{1}{(\alpha x+\beta)^2}, \quad \varphi_y = \frac{1}{(\gamma y+\delta)^2}, \quad \varphi_z = \frac{1}{(\varepsilon z+\zeta)^2}; \tag{30}$$

in which α and β are functions of y and z; γ and δ are functions of x and z; ε and ζ are functions of x and y.

These give φ the forms,

$$\frac{1}{ax+\beta}+\eta(y,z)\,,\quad\text{or}\quad \varphi=\frac{\varkappa x+\nu}{ax+\beta};$$

$$\frac{1}{\gamma y+\delta}+\theta(x,z)\,,\quad\text{or}\quad \varphi=\frac{\lambda y+o}{\gamma y+\delta};$$

$$\frac{1}{\varepsilon z+\zeta}+\iota(x,y)\,,\quad\text{or}\quad \varphi=\frac{\mu z+\pi}{\varepsilon z+\zeta};$$

where $\varkappa$, ν, a, β are functions of y and z; λ, o, γ, δ are functions of x and z; μ, π, ε, ζ are functions of x and y.

Therefore φ has the form of a fraction whose numerator and denominator are linear in x, y, and z, that is

$$\varphi=\frac{a_1x+b_1y+c_1z+d_1+e_1yz+f_1zx+g_1xy+h_1xyz}{a_4x+b_4y+c_4z+d_4+e_4yz+f_4zx+g_4xy+h_4xyz}; \tag{31}$$

similarly

$$\psi=\frac{a_2x+b_2y+c_2z+d_2+e_2yz+f_2zx+g_2xy+h_2xyz}{a_4x+b_4y+c_4z+d_4+e_4yz+f_4zx+g_4xy+h_4xyz}, \tag{32}$$

and

$$\omega=\frac{a_3x+b_3y+c_3z+d_3+e_3yz+f_3zx+g_3xy+h_3xyz}{a_4x+b_4y+c_4z+d_4+e_4yz+f_4zx+g_4xy+h_4xyz}. \tag{33}$$

It will be noticed that the denominators of φ, ψ and ω are the same, this follows from equations (11). For, supposing

$$\varphi=\frac{D(x,y,z)}{L(x,y,z)},\quad\text{and}\quad \psi=\frac{E(x,y,z)}{M(x,y,z)}, \tag{34}$$

where D, E, L, M are functions linear in x, y, and z, then the equations (11) give

$$\frac{\psi_x}{\varphi_x}=\frac{L^2}{M^2}\frac{(ME_x-EM_x)}{(LD_x-DL_x)}=P(y,z); \tag{35}$$

$$\frac{\psi_y}{\varphi_y}=\frac{L^2}{M^2}\frac{(ME_y-EM_y)}{(LD_y-DL_y)}=Q(z,x); \tag{36}$$

$$\frac{\psi_z}{\varphi_z}=\frac{L^2}{M^2}\frac{(ME_z-EM_z)}{(LD_z-DL_z)}=R(x,y). \tag{37}$$

But, since D, E, L, and M are linear in x, y, and z, the parentheses in (35), (36), (37) are themselves functions, respectively, of y, z; x, z; x, y. Therefore (35) says that L and M are the same as far as x is concerned, (36) that they have the same form relative to y, and (37) that z enters both in the same man-

ner; therefore L and M have the same form and can differ only by a constant factor. But a constant factor will not change the nature of the transformation, since it can only change the coefficients and all of them in the same ratio.

If now the forms (34), (35), (36) be substituted in the equations of condition (9) and (10) it will be seen that we must have $e_1 = f_1 = g_1 = h_1 = e_2 = f_2 = g_2 = h_2 = e_3 = f_3 = g_3 = h_3 = e_4 = f_4 = g_4 = h_4 = 0$, in order that (9) and (10) may be satisfied.

Therefore, finally, φ, ψ and ω have the forms:

$$x_1 = \varphi(x, y, z) = \frac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4}, \tag{38}$$

$$y_1 = \psi(x, y, z) = \frac{a_2x + b_2y + c_2z + d_2}{a_4x + b_4y + c_4z + d_4}, \tag{39}$$

$$z_1 = \omega(x, y, z) = \frac{a_3x + b_3y + c_3z + d_3}{a_4x + b_4y + c_4z + d_4}, \tag{40}$$

as the analytical expressions of the general projective transformation in ordinary space.

It is possible to determine the finite and infinitesimal forms of the transformation directly from its definition, namely, that it leaves the plane invariant, or more strictly the family of ∞^3 planes in space is invariant by the transformation. Thus φ, ψ, and ω might be found from the conditions that the equations $\frac{\partial^2 t}{\partial x^2} = 0$, $\frac{\partial^2 t}{\partial x \partial y} = 0$, $\frac{\partial^2 t}{\partial y^2} = 0$, the differential equations of the plane, preserve their form in the new variables x_1, y_1, z_1, but the process is much more complicated than the one used above. The method here employed may be extended to the case of n variables and the determination of the finite forms of the general projective transformation in space of n dimensions.

As before we have

$$E_j \equiv a_{j,1}x_1 + a_{j,2}x_2 + \dots + a_{j,n}x_n = 0, \quad j = 1, \dots, n-1.$$

This is a system of $(n-1)$ equations in n variables $x_1, x_2, \dots, x_n$ and hence may be written in the form:

$$x_k = a_k x_1 + b_k. \qquad k = 2, \dots, n$$

Then

$$\frac{d^2 x_k}{dx_1^2} = x_k'' = 0. \qquad k = 2, \dots, n$$

Let the transformation be represented analytically by

$$\bar{x}_i = \varphi_i(x_1, x_2, \dots, x_n). \qquad i = 1, \dots, n.$$

Then the condition of the invariance of the right line is given by

$$\bar{x}_k'' = 0 \qquad\qquad k = 2, \ldots, n.$$

for all values of x_i and x_i'. Hence the conditions:

$$\begin{aligned}
\varphi_{1,x_i}\varphi_{j,x_ix_i} - \varphi_{j,x_i}\varphi_{1,x_ix_i} &= 0, \\
\varphi_{1,x_i}\varphi_{j,x_kx_k} + 2\varphi_{1,x_k}\varphi_{j,x_ix_k} - \varphi_{j,x_i}\varphi_{1,x_kx_k} - 2\varphi_{j,x_k}\varphi_{1,x_ix_k} &= 0, \\
\varphi_{1,x_i}\varphi_{j,x_kx_l} + \varphi_{1,x_k}\varphi_{j,x_ix_l} + \varphi_{1,x_l}\varphi_{j,x_ix_k} \qquad & \\
- \varphi_{j,x_i}\varphi_{1,x_kx_l} - \varphi_{j,x_k}\varphi_{1,x_ix_l} - \varphi_{j,x_l}\varphi_{1,x_ix_k} &= 0.
\end{aligned} \tag{41}$$

$$i, j, k, l = 1, \ldots, n.$$

And, as in the preceding case,

$$\bar{x}_i = \varphi_i(x_1, x_2 \ldots, x_n) = \frac{a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n + a_{i,n+1}}{a_1x_1 + a_2x_2 + \ldots + a_nx_n + a_{n+1}} \tag{42}$$

will be found to satisfy the preceding system of partial differential equations. Geometrically the relations (42) represent the general projective point-transformation in space of n-dimensions. Analytically they assert that n functions φ_i of the variables x_i subject to the conditions (41) have the form (42).

The forms of the infinitesimal projective transformation* may be readily derived from the finite forms (38), (39), and (40). An infinitesimal point transformation is an operation by which a point is carried into the position of some point at an infinitesimal distance from the original position. Two transformations are inverse the one of the other when a point finds itself in its original position after the successive application of the two transformations, i. e. a transformation followed by its inverse gives the identical transformation.

Solving (38), (39), and (40) for x, y, and z, we have

$$x = \frac{a_1'x_1 + b_1'y_1 + c_1'z_1 + d_1'}{a_4'x_1 + b_4'y_1 + c_4'z_1 + d_4'}, \tag{43}$$

$$y = \frac{a_2'x_1 + b_2'y_1 + c_2'z_1 + d_2'}{a_4'x_1 + b_4'y_1 + c_4'z_1 + d_4'}, \tag{44}$$

$$z = \frac{a_3'x_1 + b_3'y_1 + c_3'z_1 + d_3'}{a_4'x_1 + b_4'y_1 + c_4'z_1 + d_4'}. \tag{45}$$

* The notion of a projective transformation dates back as far as Apollonius. The designation "projective" appears in Poncelet's "Traité des proprietés projectives des figures" 1822. Möbius gave the first analytical representation of the projective transformation, and in homogeneous coordinates, in "Der barycentrische Calcul" 1827. But the idea and theory of infinitesimal transformations, of which the infinitesimal projective transformation is but one type, are due to Lie.

The transformation (43), (44), (45) is the inverse of that represented by (38), (39) and (40), and has exactly the same form as the latter. Then the group of all projective transformations in ordinary space contains the inverse of every one of its transformations. The successive application of the above two transformations then gives

$$x_1 = x\,, \quad y_1 = y\,, \quad z_1 = z\,, \tag{46}$$

the identical transformation. It follows, therefore, that by proper choice of the constants (38), (39), and (40) should reduce to (46). This choice of constants is obviously expressed by the system of values below :

$$\begin{aligned} &a_1 = 1\,, \quad b_1 = 0\,, \quad c_1 = 0\,, \quad d_1 = 0\,;\\ &a_2 = 0\,, \quad b_2 = 1\,, \quad c_2 = 0\,, \quad d_2 = 0\,;\\ &a_3 = 0\,, \quad b_3 = 0\,, \quad c_3 = 1\,, \quad d_3 = 0\,;\\ &a_4 = 0\,, \quad b_4 = 0\,, \quad c_4 = 0\,, \quad d_4 = 1\,. \end{aligned}$$

Hence, since this system gives the identical transformation, to obtain the infinitesimal transformation* it is only necessary to set in (38), (39), and (40) values differing by an infinitesimal from the above. Then, if δt be an infinitesimal, and $\alpha_1, \alpha_2, \ldots$, arbitrary constants, the following are the values of the constants in the infinitesimal point transformation in three variables :

$$\begin{array}{llll} a_1 = 1 + \alpha_1\delta t\,, & b_1 = \beta_1\delta t\,, & c_1 = \gamma_1\delta t\,, & d_1 = \delta_1\delta t\,;\\ a_2 = \alpha_2\delta t\,, & b_2 = 1 + \beta_2\delta t\,, & c_2 = \gamma_2\delta t\,, & d_2 = \delta_2\delta t\,;\\ a_3 = \alpha_3\delta t\,, & b_3 = \beta_3\delta t\,, & c_3 = 1 + \gamma_3\delta t\,, & d_3 = \delta_3\delta t\,;\\ a_4 = \alpha_4\delta t\,, & b_4 = \beta_4\delta t\,, & c_4 = \gamma_4\delta t\,, & d_4 = 1 + \delta_4\delta t\,. \end{array}$$

By substituting these values in (38), (39), and (40), they give the following general expressions for the infinitesimal projective transformation in space :

$$\begin{aligned} x_1 &= \frac{x + (\alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1)\,\delta t}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t}\,,\\ y_1 &= \frac{y + (\alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2)\,\delta t}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t}\,,\\ z_1 &= \frac{z + (\alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3)\,\delta t}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t}\,. \end{aligned} \tag{47}$$

* See Page's article on "Transformation Groups," Annals of Mathematics, Vol. VIII, No. 4, p. 121.

But, as far as terms of the first order in δt,

$$\frac{1}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t} = 1 - (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t\,.$$

Therefore, as far as terms of the first order in δt,

$$\begin{aligned} x_1 &= x - (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,x\delta t + (\alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1)\,\delta t\,, \\ y_1 &= y - (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,y\delta t + (\alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2)\,\delta t\,, \\ z_1 &= z - (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,z\delta t + (\alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3)\,\delta t\,. \end{aligned} \tag{48}$$

Or,

$$\begin{aligned} x_1 &= x + \{\delta_1 + (\alpha_1 - \delta_4)\,x + \beta_1 y + \gamma_1 z - \alpha_4 x^2 - \beta_4 xy - \gamma_4 zx\}\,\delta t\,, \\ y_1 &= y + \{\delta_2 + \alpha_2 x + (\beta_2 - \delta_4)\,y + \gamma_2 z - \alpha_4 xy - \beta_4 y^2 - \gamma_4 yz\}\,\delta t\,, \\ z_1 &= z + \{\delta_3 + \alpha_3 x + \beta_3 y + (\gamma_3 - \delta_4)\,z - \alpha_4 xz - \beta_4 yz - \gamma_4 z^2\}\,\delta t\,; \end{aligned} \tag{49}$$

where α, β, γ, δ are arbitrary constants and δt an infinitesimal.

By taking the following values of the constants :

$$\begin{array}{lllll} \delta_1 = a\,, & \alpha_1 - \delta_4 = e\,, & \beta_1 = h\,, & \gamma_1 = k\,, & -\alpha_4 = n\,, \\ \delta_2 = b\,, & \alpha_2 = f\,, & \beta_2 - \delta_4 = i\,, & \gamma_2 = l\,, & -\beta_4 = p\,, \\ \delta_3 = c\,, & \alpha_3 = g\,, & \beta_3 = j\,, & \gamma_3 - \delta_4 = m\,, & -\gamma_4 = q\,, \end{array}$$

the relations (40) take the following convenient forms :

$$\begin{aligned} x_1 &= x + (a + ex + hy + kz + nx^2 + pxy + qzx)\,\delta t\,, \\ y_1 &= y + (b + fx + iy + lz + nxy + py^2 + qyz)\,\delta t\,, \\ z_1 &= z + (c + gx + jy + mz + nxz + pyz + qz^2)\,\delta t\,, \end{aligned} \tag{50}$$

the final expressions of the infinitesimal projective transformation in ordinary space. By it the coordinates x, y, z receive the increments,

$$\begin{aligned} \delta x &= x_1 - x = (a + ex + hy + kz + nx^2 + pxy + qzx)\,\delta t\,, \\ \delta y &= y_1 - y = (b + fx + iy + lz + nxy + py^2 + qyz)\,\delta t\,, \\ \delta z &= z_1 - z = (c + gx + jy + mz + nxz + pyz + qz^2)\,\delta t\,. \end{aligned} \tag{51}$$

Any arbitrary function $f(x, y, z)$ will receive the increment

$$\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial z}\,\delta z$$

by the transformation ; or

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

Now putting $\frac{\partial f}{\partial t}=Uf$ as a symbol of the operation, we have

$$\begin{aligned}Uf = {} & (a+ex+hy+kz+nx^2+pxy+qzx)\frac{\partial f}{\partial x}\\ & +(b+fx+iy+lz+nxy+py^2+qyz)\frac{\partial f}{\partial y}\\ & +(c+gx+jy+mz+nxz+pyz+qz^2)\frac{\partial f}{\partial z},\end{aligned}$$

as the general symbol of the general infinitesimal transformation in ordinary space, and this symbol defines the transformation completely. It is also noted that there are ∞^{14} infinitesimal projective transformations in ordinary space. Uf above contains fifteen arbitrary constants, but since ∂f is an arbitrary infinitesimal we may divide the expression for Uf through by one of the constants that is different from zero without altering the transformation ; hence there are only ∞^{14} infinitesimal projective transformations in ordinary space.

LEIPZIG, *November* 1, 1895.

ON THE EXPANSION OF A FUNCTION WITHOUT USE OF DERIVATIVES.*

By Prof. W. H. Echols, Charlottesville, Va.

1. Let $F(x)$ be a uniform, finite, and continuous function throughout the interval from $x = a$ to $x = c$, inclusive. If the successive derivatives are also uniform, finite, and continuous, the function will be called a Taylor's function when it can be expressed by Taylor's series throughout the interval (ac). For the present we consider $F(x)$ to be a Taylor's function.

Let

$$\int_1 F(x) = \int_a^x F(x)\,dx\,,$$

$$\int_2 F(x) = \int_a^x dx \int_a^x F(x)\,dx$$

$$= \int_a^x dx \int_1 F(x)\,,$$

and generally

$$\int_n F(x) = \int_a^x dx \int_{n-1} F(x)\,,$$

be called the successive definite integrals of $F(x)$ with respect to the base a, or briefly the successive integrals of $F(x)$.

2. If the function $F(x)$ and its first n successive integrals vanish at $x = c$, then must the function $F(x)$ vanish $n + 1$ distinct times between a and c, inclusive of c. For,

$$\int_n F(c) = \int_a^c dx \int_{n-1} F(x)$$

$$= (c - a) \int_{n-1} F(u_1) \qquad\qquad a < u_1 < c$$

$$= 0\,.$$

Therefore

$$\int_{n-1} F(x) = 0\,,$$

for $x = u_1$. But

$$\int_{n-1} F(x) = 0\,,$$

* Read before American Mathematical Society, at Springfield, Mass., Aug. 27, 1895.

for $x = c$, and

$$\int_{n-1} F(x) = \int_a^{u_1} dx \int_{n-2} F(x) + \int_{u_1}^{c} dx \int_{n-2} F(x).$$

Also

$$\int_{n-1} F(u_1) = \int_a^{u_1} dx \int_{n-2} F(x)$$

$$= (u_1 - a) \int_{n-2} F(u_2) \qquad a < u_2 < u_1$$

$$= 0.$$

Consequently,

$$\int_{n-2} F(x) = 0,$$

for $x = u_2$, and therefore must

$$\int_{u_1}^{c} dx \int_{n-2} F(x) = (c - u_1) \int_{n-2} F(u_3) \qquad u_1 < u_3 < c$$

$$= 0,$$

and we have

$$\int_{n-2} F(x) = 0,$$

for the three values of x, viz. u_2, u_3, c.

Proceeding in like manner, we show that

$$\int_{n-r} F(x) = 0,$$

for $r + 1$ values of x between a and c, including c; and finally that

$$F(x) = 0,$$

for $n + 1$ values $u_1, u_2, \ldots, u_n, c$ of x in the interval (ac).

3. By a theorem of the Differential Calculus,* when $F(x)$ is a Taylor's function in (ac), we have

$$F(x) = (x - u_1) \ldots (x - u_n)(c - x) \frac{F^{n+1}(u)}{(n+1)!};$$

and since we may write $x - u_r = \theta_r (c - a)$, and $c - x = \theta' (c - a)$, wherein the absolute values of θ_r, θ' lie between zero and unity, we have

$$F(x) = \theta^{n+1} (c - a)^{n+1} \frac{F^{n+1}(u)}{(n+1)!},$$

wherein $0 < \theta < 1$ and u lies in the interval (ac).

* See Laurent, Traité d'Analyse, t. i; or Annals of Mathematics, vol. viii, p. 74.

4. The function

$$F(x) = \frac{\begin{vmatrix} f(x), & \varphi_1(x), & \dots, & \varphi_{n+1}(x) \\ f(c), & \varphi_1(c), & \dots, & \varphi_{n+1}(c) \\ \int_1 f(c), & \int_1 \varphi_1(c), & \dots, & \int_1 \varphi_{n+1}(c) \\ \cdot & \cdot & \cdot & \cdot \\ \int_n f(c), & \int_n \varphi_1(c), & \dots, & \int_n \varphi_{n+1}(c) \end{vmatrix}}{\begin{vmatrix} \varphi_1(c), & \int_1 \varphi_2(c), & \dots, & \int_n \varphi_{n+1}(c) \end{vmatrix}}$$

vanishes, as well as do its first n successive integrals at $x = c$. Consequently, if the φ-functions are Taylor functions also,

$$f(x) = \sum_0^{n+1} A_r \varphi_r(x) + \theta^{n+1} \frac{(a-c)^{n+1}}{(n+1)!} F^{n+1}(u),$$

in which the coefficients of the φ-functions contain only definite integrals.

5. Let

$$\varphi_{r+1}(x) = \frac{1}{r!} \left[\frac{x-a}{c-a}\right]^r.$$

Then,

$$F^{n+1}(u) = f^{n+1}(u),$$

and we have

$$\frac{\begin{vmatrix} f(x), & 1, & \frac{1}{1!}\left[\frac{x-a}{c-a}\right], & \frac{1}{2!}\left[\frac{x-a}{c-a}\right]^2, & \dots, & \frac{1}{n!}\left[\frac{x-a}{c-a}\right]^n \\ f(c), & \frac{1}{0!}, & \frac{1}{1!}, & \frac{1}{2!}, & \dots, & \frac{1}{n!} \\ \frac{1}{c-a}\int_1 f(c), & \frac{1}{1!}, & \frac{1}{2!}, & \frac{1}{3!}, & \dots, & \frac{1}{(n+1)!} \\ \frac{1}{(c-a)^2}\int_2 f(c), & \frac{1}{2!}, & \frac{1}{3!}, & \frac{1}{4!}, & \dots, & \frac{1}{(n+2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{(c-a)^n}\int_n f(c), & \frac{1}{n!}, & \frac{1}{(n+1)!}, & \frac{1}{(n+2)!}, & \dots, & \frac{1}{(2n)!} \end{vmatrix}}{\begin{vmatrix} \frac{1}{0!}, & \frac{1}{2!}, & \frac{1}{4!}, & \dots, & \frac{1}{(2n)!} \end{vmatrix}} = \frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u),$$

or

$$f(x) \sum_{r=0}^{n} A_r \frac{(-1)^r}{r!} \left[\frac{x-a}{c-a}\right]^r + \frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u), \tag{i}$$

or

$$f(x) = \sum_{r=0}^{n} A_r \frac{(-1)^r}{(c-a)^r} \int_r f(c) + \frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u), \tag{ii}$$

wherein

$$A_r = \sum_{p=0}^{n} (-1)^p \frac{\Delta(r,p)}{\Delta} \frac{\int_p f(c)}{(c-a)^p}, \qquad \text{in (i)}$$

or

$$A_r = \sum_{p=0}^{n} (-1)^p \frac{\Delta(r,p)}{\Delta} \frac{1}{p!} \left[\frac{x-a}{c-a}\right]^p; \qquad \text{in (ii)}$$

and $\Delta(m, n)$ is the per-symmetric determinant

$$\Delta = \left| \frac{1}{0!}, \frac{1}{2!}, \ldots, \frac{1}{(2n)!} \right|,$$

with its mth column and nth row deleted.

Since

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots,$$

we have

$$\frac{1}{(c-a)^r} \int_r f(c) = \frac{1}{r!} \left\{ f(a) + \frac{c-a}{1!} \frac{f'(a)}{r+1} + \frac{(c-a)^2}{2!} \frac{f''(a)}{(r+1)(r+2)} + \ldots \right\},$$

which becomes evanescent as r increases without limit, as does also

$$\frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u)$$

when $n = \infty$.

The series

$$\sum_{r=0}^{\infty} \frac{1}{r!} \left[\frac{x-a}{c-a}\right]^r \quad \text{and} \quad \sum_{r=0}^{\infty} \frac{1}{(c-a)^r} \int_r f(c)$$

are absolutely convergent. If the number $[\Delta(r,p)/\Delta]_{n=\infty}$ has a finite limit, the series (i) and (ii) are convergent infinite series. Unfortunately, I have not, as yet, been able to compute these numbers, and thus put the series in practicable form.

6. Under the assumption that $[\Delta(r,p)/\Delta]_{n=\infty}$ is finite, we have

$$f(x) = \sum_{r=0}^{\infty} A_r \frac{(-1)^r}{r!} \left[\frac{x-a}{c-a}\right]^r \tag{iii}$$

throughout the interval (ac), and

$$f^r(a) = \frac{A_r}{(a-c)^r}$$

for $f(x)$ any Taylor's function.

The coefficients A_r nowhere involve derivatives, but only numbers and definite integrals. If $f(x)$ be uniform, finite, and continuous, but derivativeless throughout (ac), its successive integrals still obtain, and the numerical equivalence between the function and the series (iii) still must hold good. But the series of integral powers of $(x - a)$ represents a Taylor's function having the same *arithmetical* locus as the function $f(x)$ throughout (ac). The result, geometrically interpreted, is that we have drawn a Taylor's function curve having the same geometrical locus as the analytical function $f(x)$ throughout (ac), and the tangent to the Taylor's curve at any point x, represents the geometrical direction of the analytical curve there, which direction is analytically indeterminate.*

UNIVERSITY OF VIRGINIA, *April*, 1895.

* The possibility of the assumption that a Taylor's function could have the same geometrical locus as that of a uniform, finite, continuous, and derivativeless function, was not approved of at the reading of this note before the Society.

ON SYSTEMS OF SIX POINTS LYING IN THREE WAYS IN INVOLUTION.

By Prof. H. Maschke, Chicago, Ill.

In the following paper a system of 6 points will be studied which are in involution in three ways such that no pair of conjugate points occurs more than once. Sextuples of points of this kind are of importance for several mathematical problems. They not only occur in geometrical investigations, as for instance in the theory of multiply perspective triangles, in Clebsch's hexagon, in the metharmonic division, etc.,* but also in analytic researches.† As shown in § 1, the theory of these points is identical with the theory of the dihedron for $n = 3$. In most of the above mentioned problems those cases are of peculiar interest in which some of the anharmonic ratios that can be formed out of 4 points contained in the sextuple, have real values. To these cases the greater part of the present paper is devoted (§§ 2–5); in particular, the case in which all the 6 points are real is studied in § 5.

§ 1. The general theory.

If 6 points, defined in the plane of complex numbers by the values of a complex variable z, are in involution, there exists a linear transformation of z of period 2 producing the required correspondence of the 6 points in conjugate pairs.

If the 6 points are twice in involution in such a way that the two involutions have no pair of conjugate points in common, then the notation of the 6 points 1, 2, ..., 6 can always be arranged so that the first involution can be written

$$S:(14)\,(26)\,(35)\,, \tag{1}$$

the second

$$T:(15)\,(24)\,(36)\,. \tag{2}$$

Denoting the linear z-transformations corresponding to (1) and (2) by S and

* Besides the references given below in this paper, cf. Rosanes: Ueber Dreiecke in perspectivischer Lage. Math. Annalen, Bd. 2, p. 549; Schröter: Ueber perspectivisch liegende Dreiecke. Math. Annalen, Bd. 2, p. 553; Fuortes: Dimonstratione di due theorimi di geometria. Battaglini G. IX, p. 50; E. H. Moore: A problem suggested in the geometry of nets of curves applied to the theory of 6 points having multiply perspective relations. American Journ. Vol. 10, p. 241.

† cf. Hutchinson: On the reduction of hyperelliptic functions to elliptic functions by a transformation of the second degree. Thesis, University of Chicago, 1895.

T, respectively, we find that the transformation STS, also of period 2, produces the following involution:

$$STS : (16)\ (25)\ (34)\,. \tag{3}$$

Hence we have the theorem,

If 6 points are twice in involution in such a way that the two involutions have no pair of conjugate points in common, then they are always a third time in involution.

When S and T are combined in all possible manners, only 6 distinct transformations are obtained. Besides the identity and (1), (2), (3), we shall have the two following :—

$$ST : (123)\ (456)\,, \tag{4}$$

$$(ST)^2 : (132)\ (465)\,. \tag{5}$$

These 6 transformations as well as the 6 corresponding permutations of the 6 points form a group, the well known dihedron-group for $n = 3$.

Let $z_1, z_2, \ldots, z_6$ be the values of z in the 6 points 1, 2, ..., 6, and

$$\varphi(z) = (z - z_1)\,(z - z_2) \ldots (z - z_6)\,; \tag{6}$$

then $\varphi(z)$ admits the linear transformations of a dihedron-group, $n = 3$, i. e. if the 6 linear, homogeneous substitutions of a certain dihedron-group are applied to z_1 and z_2, $\varphi(z)$ remains unaltered, when written in the homogeneous form by putting $z = z_1 : z_2$. We will therefore call these points *six dihedron-points.*

We now describe a sphere with radius 1 and with the zero-point O of the z-plane as center, and determine a point P on this sphere by rectangular coordinates ξ, η, ζ, where the plane $\zeta = 0$ coincides with the z-plane, and the ξ-axis with the axis of real numbers. If then N denotes the point of intersection between the sphere and the positive direction of the ζ-axis, θ the angle NOP, and φ the angle between the plane NOP and the ξ-ζ plane, we have

$$\left.\begin{aligned} \xi &= \sin\theta\cos\varphi\,, \\ \eta &= \sin\theta\sin\varphi\,, \\ \zeta &= \cos\theta\,; \end{aligned}\right\} \tag{7}$$

and the projection of P from N on the z-plane is determined by

$$z = \frac{\xi + i\eta^*}{1 - \zeta} = \frac{e^{i\phi}\,.\,\sin\theta}{1 - \cos\theta}\,,$$

or

$$z = e^{i\phi}\,.\,\cot\tfrac{1}{2}\,\theta\,. \tag{8}$$

* cf. Klein, Ikosaëder, p. 32.

Let now the sphere undergo the rotations of a dihedron-group, $n = 3$. An arbitrary point a_1 on this sphere may be transferred to a_2 and a_3 by a rotation of $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ about the ζ-axis. Let a rotation of π about the ξ-axis transfer a_1 to a_4, a_2 to a_6 and a_3 to a_5. The projections of these 6 points from N are then defined by the following values of z which we will denote by capital letters:

$$\left.\begin{array}{ll} Z_1 = e^{i\phi}\cot\tfrac{1}{2}\theta\,, & Z_4 = e^{-i\phi}\tan\tfrac{1}{2}\theta\,, \\ Z_2 = \varepsilon e^{i\phi}\cot\tfrac{1}{2}\theta\,, & Z_5 = \varepsilon e^{-i\phi}\tan\tfrac{1}{2}\theta\,, \\ Z_3 = \varepsilon^2 e^{i\phi}\cot\tfrac{1}{2}\theta\,, & Z_6 = \varepsilon^2 e^{-i\phi}\tan\tfrac{1}{2}\theta\,, \end{array}\right\} \tag{9}$$

where $\varepsilon = e^{\frac{2}{3}\pi i}$.

Putting now

$$\Phi(z) = (z - Z_1)(z - Z_2)\ldots(z - Z_6)\,, \tag{10}$$

we find

$$\Phi(z) = z^6 - mz^3 + 1\,, \tag{11}$$

where

$$m = e^{3i\phi}\cot^3\tfrac{1}{2}\theta + e^{-3i\phi}\tan^3\tfrac{1}{2}\theta\,. \tag{12}$$

To this form Φ every binary sextic which remains unchanged for a dihedron-group, $n = 3$, must be reducible by a linear transformation.*

The configuration of the 6 points $Z_1, Z_2, \ldots, Z_6$ (9) consists of 2 regular triangles with the common center O; it will be called the *normal-position* of 6 dihedron-points.

We have now the theorem,

If 6 *points are twice in involution in such a way that the two involutions have no pair of conjugate points in common, then they form a sextuple of dihedron-points and can always be transformed by a linear substitution into the normal-position of* 6 *dihedron-points.*

Those 6 linear transformations which convert the 6 dihedron-points of the normal-position into themselves, are given by

$$z' = z\,,\quad z' = \varepsilon z\,,\quad z' = \varepsilon^2 z\,, \tag{13}$$

and

$$z' = 1/z\,,\quad z' = \varepsilon/z\,,\quad z' = \varepsilon^2/z\,. \tag{14}$$

The three latter are of period 2 and represent the three involutions (1), (2), and (3). The double points of these involutions are obtained by putting $z' = z$ in (14), which gives $z = \pm 1, \pm \varepsilon, \pm \varepsilon^2$. Hence

The double points of the 3 *involutions of* 6 *dihedron-points in the normal-position are entirely independent of the special position of the point* a_1 *on the sphere, i. e. of* θ, φ, *and* m; *they represent on the equator (intersection between sphere and z-plane) a regular hexagon:* $z^6 - 1 = 0$.

* cf. Klein, Ikosaëder, p. 49, and Bolza: On binary sextics with linear transformations into themselves, American Journal, Vol. X, p. 49.

§ 2. Anharmonic ratios.

The anharmonic ratios of the quadruples of points contained in the dihedron-sextuple have, in general, complex values. Of special interest are those cases in which some of these ratios are real, and, in particular, harmonic.

The anharmonic ratio (12) (34) of 4 points z_1, z_2, z_3, z_4 in the complex plane is defined as

$$(12)\,(34) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}. \tag{15}$$

The necessary and sufficient condition for the reality of this anharmonic ratio is that the 4 points lie on a circle, and in this case the above defined anharmonic ratio is identical with the anharmonic ratio of 4 points on a circle (conic section) as defined in projective geometry.*

The projection of any circle of the sphere from N on the z-plane is again a circle, and a circle on the z-plane remains a circle if any linear transformation is applied.

We have therefore to find out those cases in which 4 of the 6 points a_1, $a_2, \ldots, a_6$ on the sphere lie on a circle (are concyclic). Unless all the 6 points lie on the equator, the circle passing through the 3 points a_1, a_2, a_3 on the upper hemisphere cannot contain any one of the 3 other points. A quadruple of concyclic points can therefore consist only of 2 points of the upper and 2 points of the lower hemisphere. An easy geometrical consideration gives then the 3 following possibilities for the existence of concyclic quadruples :—

1. The points 1, 2, 3, of the upper hemisphere lie perpendicularly (with respect to the equator) above the points 4, 5, 6 (in this order) of the lower hemisphere, corresponding to $\varphi = 0$, or $\varphi = \pi$ (θ arbitrary). The 6 points form a right prism.

2. The 2 triangles 123 and 456 lie symmetrically to each other, corresponding to $\varphi = \pm \frac{1}{6}\pi, \pm \frac{1}{2}\pi, \pm \frac{5}{6}\pi$ (θ arbitrary). The 6 points form an oblique octahedron.

3. All the 6 points lie on the equator, corresponding to $\theta = \frac{1}{2}\pi$ (φ arbitrary).

In the two cases, 1 and 2, the 3 quadruples (1245), (2356), (3164) lie each on a circle. The anharmonic ratios of these 3 quadruples taken in corresponding orders are evidently, as the figure on the sphere shows, equal, also in the general case without any specification of the values of φ and θ. Denoting

$$(15)\,(42) = (26)\,(53) = (34)\,(61) = \lambda\,, \tag{16}$$

* cf. Klein : Einleitung in die geometrische Functionentheorie. Autographirte Vorlesung, 1880-81, pp. 42 and 43 With regard to the geometrical interpretation of anharmonic ratios cf. Moebius : Die Theorie der Kreisverwandtschaft. Crelles Journal, Bd. 52, or Gesammelte Werke, Bd. II ; and Wedekind : Beiträge zur geometrischen Interpretation binärer Formen. Math. Annalen, Bd. 9.

we find from (9) and (15)

$$\lambda = -\tfrac{1}{3}\left(e^{i\phi}\cot\tfrac{1}{2}\theta - e^{-i\phi}\tan\tfrac{1}{2}\theta\right)^2. \tag{17}$$

Eliminating $e^{i\phi}\cot\frac{1}{2}\theta$ between this equation and (12) we obtain

$$m^2 = (4 - 3\lambda)(1 - 3\lambda)^2, \tag{18}$$

or

$$\lambda(1-\lambda)^2 + \tfrac{1}{27}(m^2 - 4) = 0.$$

We now investigate the 3 cases pointed out above, in succession.

§ 3. The 6 dihedron-points on the sphere form a right prism.

Since we have in this case $\varphi = 0$ or $\varphi = \pi$, it follows that

$$m = \pm\left(\cot^3\tfrac{1}{2}\theta + \tan^3\tfrac{1}{2}\theta\right). \tag{19}$$

The form $\Phi(z) = z^6 - mz^3 + 1$ represents therefore this case, if m takes any real value between $+2$ and $+\infty$ or between -2 and $-\infty$.

It is now sufficient to consider only the case $\varphi = 0$. We find from (17)

$$\lambda = -\tfrac{1}{3}\left(\cot\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta\right)^2,$$

which reduces to

$$\lambda = -\tfrac{4}{3}\cot^2\theta. \tag{20}$$

The 3 circles in which the 3 rectangular faces of the prism meet the sphere, intersect each other at equal angles. Denoting this angle by α, we have to find the connection between α, θ and λ.

The angle α remains unchanged by the projection from the sphere on the z-plane, and also by any linear transformation of the z-plane. To the normal-position representing our case (see Fig. 1) we apply a linear transformation

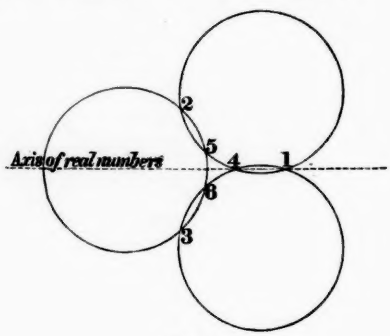


Fig. 1.

that converts the two circles (1452) and (1463) into straight lines, while the axis of real numbers remains unchanged. This is done by the transformation

$$z' = \frac{1}{z - Z_1}. \tag{21}$$

The point 1 is transferred to infinity and we obtain Fig. 2.

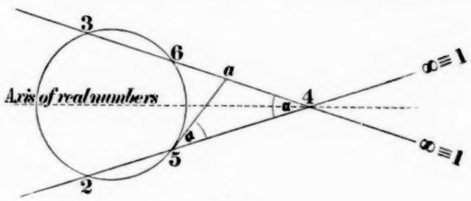


FIG. 2.

The connection between the anharmonic ratio λ (16) and the angle α can now be found in the following way :—

Draw the tangent to the circle in point 5 meeting the line 46 in point a. Then $\angle\, 45a = \alpha$. The anharmonic ratio of the pencil obtained by joining point 5 to the 4 points 2, 6, 5, 3 on the circle is also $= \lambda$. The intersection of the 4 rays of this pencil with the line 13 gives the points 4, 6, a, 3. Thus we have

$$\lambda = (46)\,(a3) = (34)\,(61)\,.$$

Denoting the distances $4a = r$, $46 = s$, $43 = t$, taken positive in the direction from 4, this double-equation becomes

$$\frac{r}{r-s} : \frac{t}{t-s} = \frac{s-t}{s} = \lambda\,,$$

and by elimination of t,

$$\frac{r}{s} = \frac{2-\lambda}{1-\lambda}.$$

On the other hand the distance 45 also $= s$, and triangle $5a4$ is isosceles ; whence $\cos\alpha = s/2r$.

Thus we obtain the required connection between α and λ,

$$\cos\alpha = \frac{2-\lambda}{2-2\lambda}, \tag{22}$$

$$\lambda = \frac{2(\cos\alpha - 1)}{2\cos\alpha - 1}. \tag{23}$$

From (20) and (23) we find

$$\cos a = \frac{3 - \cos^2\theta}{3 + \cos^2\theta}, \tag{24}$$

$$\cos\theta = \sqrt{3} \,.\, \tan\tfrac{1}{2}a\,. \tag{25}$$

Equation (24) shows that a varies only between 0 and $\frac{1}{3}\pi$, which means that the curvilinear triangle 456 lies always outside the 3 circles ; while in the case $a > \frac{1}{3}\pi$, which is to be considered in § 4 (cf. Fig. 5), the triangle 456 lies inside each circle, the case $a = \frac{1}{3}\pi$ representing the limiting case, where the triangle is reduced to a point.

We have now obtained the following result :

The 6 points of intersection of any 3 circles meeting each other at equal angles a, where $a < \frac{1}{3}\pi$, constitute 6 dihedron-points ; they can always be linearly transformed into the normal-position obtained by projection of a right prism.

If $\lambda = -1$, the 6 points are harmonically divided in the following remarkable manner :

(15) is divided harmonically by (24) ,

(26) . . . . (35) ,

(34) . . . . (16) .

In this special case we have

$$\cos a = \tfrac{3}{4}\,, \quad \cos\theta = \sqrt{\tfrac{3}{7}}\,, \quad m = 4\sqrt{7}\,.$$

The rectangular faces of the right prism on the sphere are squares. It is this sextuple which I have called in a previous paper* a system of 6 *metharmonic* points. We see then that *the points of intersection of any 3 circles meeting at equal angles a, where $\cos a = \frac{3}{4}$, form a system of metharmonic points.*

§ 4. The 6 dihedron-points on the sphere form an oblique octahedron.

We have in this case $\varphi = \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{5}{6}\pi$, hence

$$m = \pm\, i\,(\cot^3\tfrac{1}{2}\theta - \tan^3\tfrac{1}{2}\theta)\,.$$

The form $\Phi(z) = z^6 - mz^3 + 1$ represents, therefore, this case, when m is purely imaginary.

* H. Maschke : Ueber eine merkwürdige Configuration gerader Linien im Raume. Math. Annalen, Bd. 36, p. 200, and Nachrichten d. Königl. Gesellschaft d. Wissenschaften zu Göttingen, 1889, p 386.

Without loss of generality we may consider the case $\varphi = \frac{1}{2}\pi$ only. The position of the 6 points appears from Fig. 3 representing the perpendicular

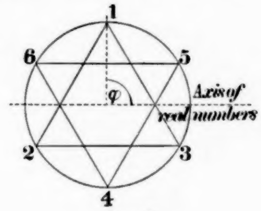


FIG. 3.

projection on the z-plane.

From (17) we find

$$\lambda = + \tfrac{1}{3}(\cot \tfrac{1}{2}\theta + \tan \tfrac{1}{2}\theta)^2,$$

or

$$\lambda = \tfrac{4}{3}/\sin^2\theta. \qquad (26)$$

Denoting, again, the common value of the angles at which the 3 circles containing the points (1245), (2356), (3164) meet each other by α, we obtain the connection between α and λ (16) in exactly the same way from Fig. 4, as the corresponding connection in § 3 had been found from Fig. 2. The formulæ

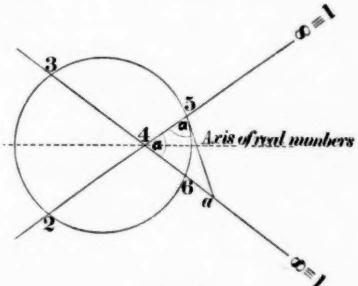


FIG. 4.

(22) and (23) hold, therefore, in this case also. Furthermore, this figure is deduced by a linear transformation similar to (21) from the normal-position of the case presented in Fig. 5.

Comparing (23) and (26) we find

$$\cos\alpha = \frac{3\cos^2\theta - 1}{3\cos^2\theta + 1},$$

$$\cos\theta = \frac{1}{\sqrt{3}}\cot\tfrac{1}{2}\alpha.$$

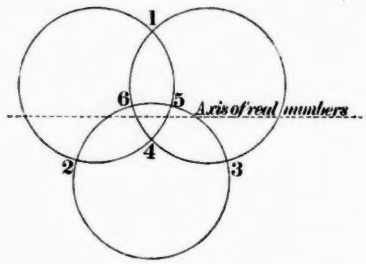


FIG. 5.

Consequently the angle α varies between $\frac{1}{3}\pi$ and π, and the anharmonic ratio λ between $+\frac{4}{3}$ and $+\infty$. Thus we have this result:—

The 6 *points of intersection of any* 3 *circles meeting each other at equal angles* α, *where* $\frac{1}{3}\pi < \alpha < \pi$, *constitute* 6 *dihedron-points; they can always be linearly transformed into the normal-position obtained by projection of an oblique octahedron.*

If $\lambda = 2$, the 6 points lie harmonically in such a way that every one of the 3 pairs (14), (25), (36) is harmonically divided by every other pair. In this special case we have

$$\alpha = \tfrac{1}{2}\pi, \quad \cos\theta = \frac{1}{\sqrt{3}}, \quad m = 5\sqrt{-2},$$

and the 6 points form on the sphere a *regular* octahedron.

§ 5. THE 6 DIHEDRON-POINTS OF THE SPHERE LIE ALL ON THE EQUATOR.

In this case we have $\theta = \frac{1}{2}\pi$ and therefore $m = 2\cos 3\varphi$. The form $\Phi(z) = z^6 - mz^3 + 1$ represents this case when m is real and varies between -2 and $+2$.

We transform the equator-circle into the axis of real numbers so that all the six points are real, and assume now, as we are only concerned with projective properties, the six points on a conic section (circle).

The lines joining the conjugate points of an involution meet in one point, the center of involution. Thus we obtain, corresponding to the 3 involutions (1), (2), (3) the 3 centers C_1, C_2, C_3 (see Fig. 6).

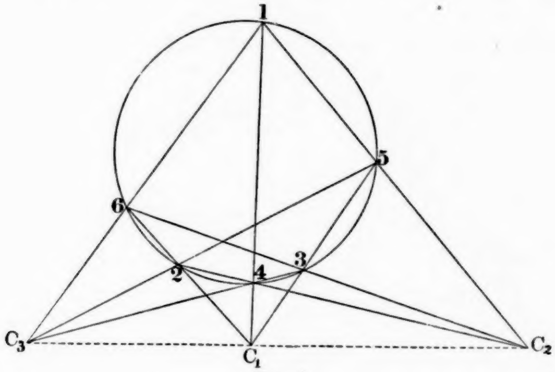


FIG. 6.

These 3 centers lie on one straight line, the Pascal-line of the hexagon 153426.

Furthermore, we see that the two triangles 123 and 456 are in perspective in 3 ways with C_1, C_2, C_3, as centers of perspective. Hess has studied in detail the case of two triply perspective triangles whose centers of perspective lie on a straight line*; he does not, however, mention that the 6 vertices of the triangles lie on a conic section, which is an immediate consequence of the converse of Pascal's theorem.

Varying the one parameter of the figure which is at our disposal—it depends on the value of φ in our normal-figure—by keeping for instance the 3 points 1, 4, 5 fixed and varying only the point 3 which determines the rest, we can find a position where the 3 lines (14), (25), (36) meet in one point. We then obtain a fourth involution (14), (25), (36), and the two triangles are in 4 ways in perspective. This case—two triangles that are perspective in 4 ways with 3 centers of perspective on a straight line—has also been studied by Hess.†

The corresponding normal-figure consists simply of a regular hexagon ($\varphi = \frac{1}{2}\pi$, $m = 0$, $\Phi(z) = z^6 + 1$); the additional center of involution is given by the center of the circle. This configuration is projectively unique; it represents the maximum number of involutions that can occur with 6 points, and also the maximum number of harmonic divisions, viz. six.

Reverting once more to Fig. 6 (φ arbitrary), let us draw tangents from C_1, C_2, C_3, to the circle. The 6 contact points represent the double points of the

* Edmund Hess "Beiträge zur Theorie der mehrfach perspectiven Dreiecke und Tetraeder." Mathemat. Annalen, Bd. 28, pp. 186 et seq.

† l. c., pp. 188 and 198.

3 involutions (1), (2), (3), and as such they form in the normal-position a regular hexagon, as shown before. It follows, then, *that these* 6 *contact points lie in involution in* 4 *ways.*

We now ask how the 6 real dihedron-points on the conic section (circle) must lie in order that some of the anharmonic ratios may become harmonic. The normal-figure shows that, besides the case of the regular hexagon, this can only occur in such a way that (12) is divided harmonically by (45), (23) by (56), and (31) by (64); or so that (12) is divided by (56), etc. The latter case being essentially the same as the first, it suffices to consider the first only.

Putting in (17) $\theta = \frac{1}{2}\pi$, we obtain,

$$\lambda = (15)(42) = \tfrac{4}{3}\sin^2\varphi\,.$$

The value $\lambda = \frac{1}{2}$ gives harmonic division, whence

$$\cos\varphi = \tfrac{1}{4}\sqrt{10}\,,$$

and from (18)

$$m = \tfrac{1}{4}\sqrt{10}\,.$$

Let now the points 1, 2, . . . , 6 (see Fig. I) be any such system of real dihedron-points on a conic section. If we draw tangents in the points 1, 2, 3 to the conic section, determining the triangle A_1, A_2, A_3, then the lines joining the points 5–6, 6–4, 4–5, must meet the points A_1, A_2, A_3, respectively. Similarly the lines joining the points 2–3, 3–1, 1–2 must pass through the vertices A_4, A_5, A_6 of the triangle formed by the tangents in the points 4, 5, 6. Thus we are led to a known problem proposed by Clausen* and solved by Moebius† : *Given a triangle with an inscribed conic section (circle). To find a second triangle that is inscribed to the conic section (circle) and circumscribed to the given triangle.*

The configuration of this sextuple on the conic section is very closely connected with the configuration of a so-called *Clebschian hexagon.*‡ A Clebschian hexagon consists of 6 points C_1, C_2, . . . , C_6 (see Fig. II) situated in the plane in such a way that every one of the 10 pairs of triangles which can be formed out of the 6 points C, is in 4 ways in perspective. The 40 centers of perspective thus obtained are reduced to 10 (the points E_1, E_2, E_3, A_1, A_2, A_3, A_4, A_5, A_6 in Fig. II), since every one of them is a center of perspective for 4

* Crelles Journ., Bd. 4, p. 391.

† Moebius : Barycentrische Lösung der Aufgabe des Herrn Clausen. Crelle's Journ., Bd. 5, and Gesammelte Werke, Bd. I, p. 487.

‡ Clebsch. Math. Annalen, Bd. 4, pp. 284 and 345. This hexagon was more fully investigated by Schröter : Das Clebsch'sche Sechseck. Math. Annalen, Bd. 28, from which paper the above given principal properties are taken, and by Hess, l. c. § 6 under the name : Das zehnfach Brianchon'sche Sechseck. Cf. also : Klein, Ikosaeder, pp. 216–218, and Lindemann, Vorlesungen über Geometrie, II, 578.

FIG. I.

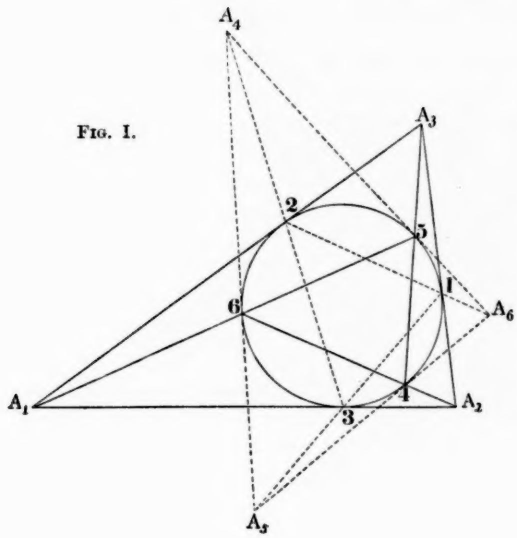
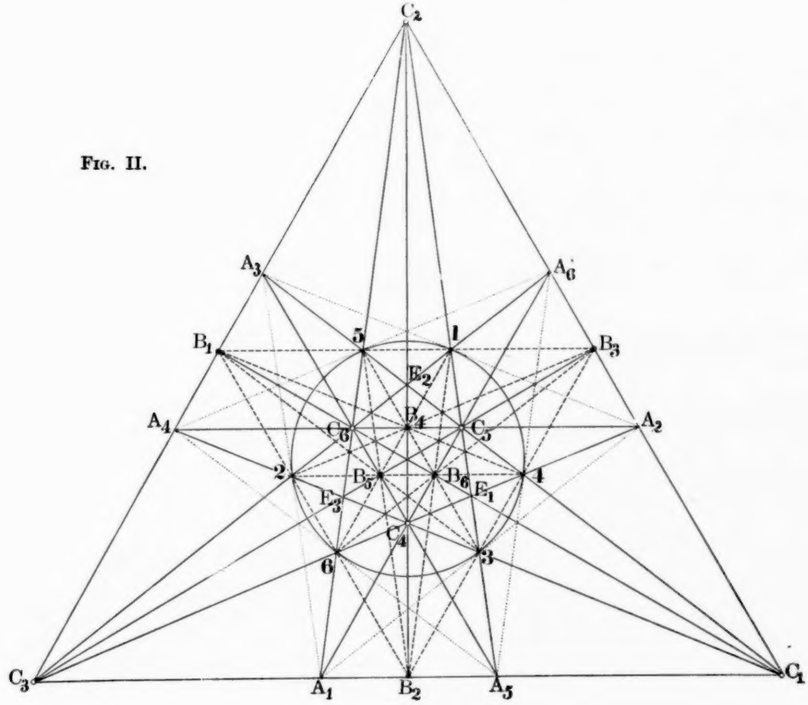

A4
A3
2
5
1
6
A6
4
A1
3
A2
A5

FIG. II.


C2
A3
A6
B1
5
1
B3
E2
A4
C6
B4
C5
A2
2
B5
B6
E1
4
E3
C4
6
3
C3
A1
B2
A5
C1

pairs of triangles. Through every center of perspective there pass, therefore, 3 of the 15 sides of the Clebschian hexagon. Besides there are 15 secondary points in which the 15 sides of the hexagon meet each other only by twos. Two centers of perspective lie on each side of the hexagon. If we form a triangle out of any 3 vertices of the hexagon, then the 6 centers of perspective situated on the 3 sides of this triangle, lie on a conic section F. The 15 secondary points can be arranged as 5 triangles in such a way that every side of each triangle is also a side of the Clebschian hexagon. Then the 6 vertices of any 2 of these 5 triangles lie on a conic section Φ.

The connection between the 6 dihedron-points 1, 2, ..., 6 in Fig. I and the Clebschian hexagon (see Fig. II, where the points corresponding to those in Fig. I are denoted by the same numbers or letters) is now simply this:

1. The dihedron-points 1–6 constitute 6 centers of perspective lying on a conic section F of a Clebschian hexagon whose vertices $C_1, C_2, \ldots, C_6$ are obtained as the points of intersection of the lines joining the points:

$$2, 3 \text{ and } 4, 5 : C_1, \quad 2, 3 \text{ and } 4, 6 : C_4,$$

$$3, 1 \text{ and } 5, 6 : C_2, \quad 3, 1 \text{ and } 5, 4 : C_5,$$

$$1, 2 \text{ and } 6, 4 : C_3, \quad 1, 2 \text{ and } 6, 5 : C_6.$$

2. The dihedron-points 1–6 constitute 6 secondary points lying on a conic section Φ of a Clebschian hexagon whose vertices $B_1, B_2, \ldots, B_6$* are obtained as the points of intersection of the lines joining the points

$$1, 5 \text{ and } 2, 6 : B_1, \quad 1, 6 \text{ and } 3, 5 : B_4,$$

$$2, 6 \text{ and } 3, 4 : B_2, \quad 2, 4 \text{ and } 1, 6 : B_5,$$

$$3, 4 \text{ and } 1, 5 : B_3, \quad 3, 5 \text{ and } 2, 4 : B_6.$$

The configuration of our points 1, 2, ..., 6 is identical with that special case of two triply perspective triangles enumerated by Hess under 2 (l. c., p. 188), as can be demonstrated by referring to Hess's triangle of reference. Hess himself has shown only (l. c., p. 209) how this figure is connected with the Clebschian hexagon $B_1, \ldots, B_6$. The connection with the hexagon $C_1, \ldots, C_6$ follows without much difficulty by a closer study of the Clebschian configuration. The relation between the two Clebschian hexagons $C_1, \ldots C_6$ and $B_1, \ldots B_6$ is given by the following theorem: *Such* 6 *secondary points as lie*

* In Fig. II the sides of the Clebschian hexagon $C_1, \ldots, C_6$ are represented by continuous lines, those belonging to the hexagon $B_1, \ldots, B_6$ by broken lines, except the 3 lines B_1C_1, B_2C_2, B_3C_3 which are common to both. The two circumscribed triangles $A_1A_2A_3$ and $A_4A_5A_6$ are given by dotted lines.

on 3 sides of a Clebschian hexagon intersecting in a center of perspective, form again a Clebschian hexagon.

Finally it may be remarked that the 6 vertices $A_1, A_2, \ldots, A_6$ of the two triangles circumscribed to the conic section in Moebius figure (see Fig. I) constitute 6 centers of perspective lying on a conic section F of a Clebschian hexagon (see Fig. II). That these points lie on a conic section, has already been found by Moebius. We now see, moreover, that these points constitute again a system of real dihedron-points of the same character as the original points 1, 2, ..., 6.

UNIVERSITY OF CHICAGO, *December* 31, 1895.

NOTE ON INFINITE DETERMINANTS.

By Mr. Eugene H. Roberts, Lonoke, Ark.

1. Definitions.

Let us consider a doubly infinite series of quantities A_{ik} $(i, k = -\infty, \ldots, +\infty)$, and as is usual with such series we shall suppose the quantities arranged over a plane, so that with any point (i, k) of the plane the quantity A_{ik} is associated; the term A_{00} will be at the origin, and the terms around it will be arranged as follows:

$$
\begin{array}{ccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & A_{3,-3} & A_{3,-2} & A_{3,-1} & A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & \cdots \\
\cdots & A_{2,-3} & A_{2,-2} & A_{2,-1} & A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & \cdots \\
\cdots & A_{1,-3} & A_{1,-2} & A_{1,-1} & A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} & \cdots \\
\cdots & A_{0,-3} & A_{0,-2} & A_{0,-1} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots \\
\cdots & A_{-1,-3} & A_{-1,-2} & A_{-1,-1} & A_{-1,0} & A_{-1,1} & A_{-1,2} & A_{-1,3} & \cdots \\
\cdots & A_{-2,-3} & A_{-2,-2} & A_{-2,-1} & A_{-2,0} & A_{-2,1} & A_{-2,2} & A_{-2,3} & \cdots \\
\cdots & A_{-3,-3} & A_{-3,-2} & A_{-3,-1} & A_{-3,0} & A_{-3,1} & A_{-3,2} & A_{-3,3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Let us denote by D_1 the determinant indicated by the dotted lines, that is the determinant composed of the quantities A_{ik} $(i, k = -1, 0, +1)$; by D_2 the determinant indicated by the light lines; and by D_3 the determinant indicated by the heavy lines; and in general let us represent by D_m the determinant composed of the quantities A_{ik} $(i, k = -m, \ldots, +m)$, arranged as indicated in the diagram.

Now if as m increases indefinitely, the determinant D_m has a definite limit D, we say that the infinite determinant D_∞ composed of the quantities A_{ik} $(i, k = -\infty, \ldots, +\infty)$ arranged as indicated in the diagram, is convergent, and has D for its value; or, to state the same thing in a form which will be of more use subsequently, the infinite determinant D_∞ is convergent if, for any

positive quantity δ given in advance, we can find a positive integer n' such that for any value of n greater than n'

$$|D_{n+p} - D_n| < \delta \tag{1}$$

whatever the value of p. If no such number n' exists, the determinant is divergent.

The *principal diagonal* of the determinant D_∞ is the group of elements arranged along the straight line passing through the origin and making an angle of 45° with the axis of x; it is composed of the quantities A_{ii} ($i = -\infty, \ldots, +\infty$). The *line* i is composed of the elements grouped along the straight line passing through the point $(i, 0)$, and parallel to the axis of x; similarly the *column* k is composed of the elements grouped along the straight line passing through the point $(0, k)$, and parallel to the axis of y. The element A_{00} is called the *origin* of the determinant; and any element A_{ik} is said to be *diagonal* or *non-diagonal* according as we have $i = k$, or $i \gtrless k$.

Any diagonal element might be taken as the origin, and corresponding to it we would have a new set of determinants D_m; any line of elements parallel to that already chosen might be taken for the principal diagonal, and any element in it might be selected as origin; we thus see that having given a doubly infinite series of quantities, we can form ∞^2 different infinite determinants, and nothing justifies us in stating *à priori* that all of these infinite determinants are convergent or divergent at the same time; or, if they are all convergent, that they have the same value. We thus see that in general an infinite determinant has a perfectly definite meaning only when we know its principal diagonal and its origin.

It is interesting to note that the different infinite determinants considered above are derived from a given arrangement of the terms of the doubly infinite series; to any new arrangement of the terms of this series, there corresponds a new set of ∞^2 different determinants.

2. Historical summary.

Infinite determinants were first defined and used by Hill in a paper entitled "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon."* In the discussion of this problem Hill met with an infinite system of linear equations involving an infinite number of unknown quantities; the consideration of such a system of equations naturally suggested the possibility of forming and using an infinite determi-

* This paper was first published in this country in 1877, and afterwards appeared in Acta Mathematica, t. 8.

nant; Hill accordingly defined a convergent determinant, his definition being substantially that given above, and applied the determinants thus defined to the solution of the system of equations under consideration. Thus the discovery of infinite determinants, like that of so many other mathematical functions, was suggested by the consideration of a special problem which demands such functions for its solution.

Infinite determinants having thus been defined and used in the solution of a special problem, naturally the next and most important step was that of investigating such functions from a purely theoretical standpoint. This was done by Poincaré in a paper entitled "Sur les determinants d'ordre infini."* Poincaré's investigations were limited to determinants satisfying the following conditions: (a) the diagonal elements are all equal to unity, and (b) the sum of the non-diagonal elements is absolutely convergent.

Helge von Koch was the next mathematician to turn his attention to infinite determinants; his results were published in two papers entitled, respectively, "Sur une application des déterminants infinis à la théorie des équations differentielles linéaires," and "Sur les déterminants infinis et les équations differentielles linéaires."† The determinants studied and used by von Koch are more general than those investigated by Poincaré; like the latter, they satisfy condition (b), but condition (a) is generalized so as to include all infinite determinants in which the product of the diagonal elements is absolutely convergent; infinite determinants satisfying these modified conditions are said to be of the *normal form*, and it is seen that the determinants studied by Poincaré form a special class of determinants of the normal form.

In the second paper cited above von Koch shows that a certain class of infinite determinants enjoys the same properties as determinants of the normal form; this class of determinants includes all that satisfy the following conditions: (a) the product of the diagonal elements is absolutely convergent; (b) there exists a series of quantities x_k ($k = -\infty, \ldots, +\infty$) such that the doubly infinite series

$$\Sigma_i \Sigma_k A_{ik} \frac{x_i}{x_k}$$

is absolutely convergent.

Von Koch, while investigating the subject from a theoretical standpoint, did so for the special purpose of applying his results to a particular problem; and for that reason his results lack that generality which is to be desired. In the introduction to his second paper he states: "Dans ce qui suit nous nous bornerons à développer ce qui paraît nécessaire pour pouvoir appliquer l'in-

* Bulletin de la Société mathématique de France, t. 14.

† Acta Mathematica, ts. 15, 16.

strument nouveau d'une manière absoluement rigoureuse au problème que nous nous sommes proposé."*

Craig has also done some work in connection with the application of infinite determinants to differential equations; while his results have been used in his course of lectures on differential equations delivered at Johns Hopkins University, his work has not as yet been published.

In the present paper an attempt is made to place the discussion of infinite determinants upon a slightly new basis, in the hope of giving greater simplicity to the reasoning employed, and at the same time more generality to the theorems established.

3. Expression of an infinite determinant as an infinite series, the terms of which are infinite products.

Returning to the definition of a convergent determinant embodied in the inequality (1), let us push this definition to its legitimate end, and thereby get a clearer view of the problem before us.

We must determine exactly what is meant by saying that the determinants D_m tend towards a limit as m increases indefinitely.

Let us take as a simple illustration the determinant,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 .$$

The configuration of letters, numbers, and lines written on the left hand side of the sign of equality is called a determinant, and is simply a convenient symbol used to represent the algebraic function written on the right hand side; this is the case with all determinants, so that if we speak of a set of determinants D_m as tending towards a limit as m increases indefinitely, we mean that the functions represented by these determinants tend towards a limit as m increases indefinitely.

We must carefully examine the nature of the function represented by a determinant; the function represented by D_m is the algebraic function consisting of $(2m + 1)!$ terms, half of which are positive, and the other half negative, each term consisting of the product of $2m + 1$ of the quantities A_{ik}, every possible combination being taken subject to the condition that one and only one element must be taken from any line (or column) for a given term. Certain conventions are adopted to determine the sign of any term.

* In the Comptes Rendus for Jan. 21, 1895, there is an article by von Koch entitled, "Sur la convergence des déterminants d'ordre infini et des fractions continues." Unfortunately I have not yet been able to consult this article.

As m increases indefinitely the number of the terms of such a function increases indefinitely, and the number of elements appearing in each term increases indefinitely; so that in the limiting case, an infinite determinant represents an infinite series of terms, some positive and some negative, each term being the product of an infinite number of elements, one and only one being taken from each line (or column) for any term.

In the case of the function represented by a finite determinant, the order of the terms, as well as that of the elements in any term, is a matter of indifference; this is not so with an infinite determinant, however, for in general the value of an infinite sum or an infinite product is dependent upon the order of the terms of the sum, or that of the factors of the product. Consequently, having given an infinite determinant of specified origin and principal diagonal, it is not only necessary to adopt some convention by which to determine the sign of any term, but it is also necessary to adopt conventions that will fix the order of the terms in any determinant and the order of the elements in each term.

Rules sufficient to determine the sign of any term may be formed as an extension of those adopted for finite determinants. In determining the sign of any term the elements are all supposed to be positive; of course any odd number of negative elements in a term would change its sign, but these changes adjust themselves by purely algebraic principles.

The sign of the principal term is determined by the following rule:—

(1) The term $\overset{+\infty}{\underset{-\infty}{\Pi_i}} A_{ii}$, composed of the diagonal elements written in the order in which they occur in the infinite determinant, is positive.

In this term both suffixes occur in their natural order; other terms may be formed from this principal term by permuting either set of suffixes; we shall in all that follows suppose the first set of suffixes to remain fixed, and form the different terms by permuting the second set of suffixes; this amounts to adopting a convention for determining the order of the elements in any term. This being agreed upon, the sign of any term is determined by the following rule:—

(2) Any interchange of two suffixes of the second set, the suffixes of the first set retaining their order, alters the sign of the term.

The convention that the elements in any term be written so that the first set of suffixes are in their natural order is a natural one; in fact it is generally observed in regard to finite determinants, for if we have a determinant the principal term of which is $a_1b_2c_3d_4 \ldots e_n$, in writing any other term it is usual to write the letters in their natural order, the permutations being made among the suffixes; while any violation of this rule in the case of a finite determinant

will not alter the value of the determinant, this is not in general true for an infinite determinant and the adoption of such a rule is a necessity.

Having thus adopted a rule which is sufficient to fix the order of the elements in the infinite product of any term, it remains to consider the question of the order of the terms of the infinite series.

Three conventions in regard to the arrangement of the terms of an infinite determinant suggest themselves naturally; they are: (1) to write the principal term first; (2) to choose the order of the terms so that they are alternately positive and negative; (3) to choose the order of the terms so that there is a symmetrical arrangement in regard to the principal term, for example, if the fourth term to the right of the principal term is obtained by interchanging $-m$ and $+n$, the fourth term to the left is to be obtained by interchanging $+m$ and $-n$. Rule (3) is easily seen to be in harmony with (2). Even though we adopt these three conventions, the order of the terms is still undetermined, and there is open to us for choice an infinite number of rules any one of which is sufficient to fix the order of the terms. This being the case, the question naturally arises whether there exists a class of determinants the values of which are independent of the particular law chosen to fix the order of their terms. This question is answered very easily, and furnishes the most natural basis for dividing infinite determinants into two general classes according to the nature of the series represented by the infinite determinant.

However, before proceeding to the consideration of this classification of infinite determinants, it may be well to give a brief résumé of the results arrived at in this article. We have shown that an infinite determinant represents an infinite series, the terms of which are infinite products; we have formulated rules sufficient to determine the sign of any term and the order of the factors in the infinite product constituting any term. We have also seen that any one of an infinite number of possible rules would be sufficient to determine the order of the terms in the series.

4. Distinction between the two kinds of infinite determinants.

Before classifying determinants according to the nature of the series represented by them, it is necessary to call attention to a distinction which must be made in regard to the infinite determinants we are to consider. Since the number of the terms of the determinant D_m is $M = (2m + 1)!$, if we have $m = \infty$, M becomes infinite of an order higher than the first, and consequently the function represented by such a determinant is not an ordinary infinite series, but is one of an order higher than the first. This class of infinite determinants for which $m = \infty$, we shall call infinite determinants of the first kind.

However, if instead of considering m as approaching infinity we consider M, the number of terms of the function represented by D_m, as approaching infinity, then for $M = \infty$, the series which the determinant represents is an ordinary infinite series. Since M is finite for any finite value of m, however large, if we have $M = \infty$, m must also be infinite, though necessarily of an order lower than the first. Infinite determinants for which $M = \infty$ we shall call infinite determinants of the second kind. They are infinite determinants in just as strict a sense of the word as are those of the first kind, but in order to be perfectly rigorous it is necessary to make the distinction. In what follows we shall confine our investigations to infinite determinants of the second kind.*

5. Semi-convergent and absolutely convergent determinants.

Since the consideration of an infinite determinant has been reduced to the consideration of an infinite series, it is natural to divide infinite determinants into classes according to the nature of the series represented by such determinants.

The first and most important classification of series is that of dividing them into the two classes of convergent and divergent series; and following this classification we may say that an infinite determinant is convergent or divergent according as it represents a convergent or divergent series. [This is not a new definition of convergence, but simply a new way of stating the one already given.]

Convergent determinants may be subdivided into two classes, semi-convergent and absolutely convergent determinants, according as they represent semi-convergent or absolutely convergent series. Every convergent determinant must belong to one of these two classes, and no determinant can belong to both classes.

Absolutely convergent determinants constitute by far the most interesting class of determinants, for the value of the series represented by such a determinant being independent of the order of its terms, we conclude that *an absolutely convergent determinant has the same value, whatever the law adopted to fix the order of the terms in the series represented by it;* and consequently in the study of such determinants we are freed from the necessity of adopting a law to fix the order of the terms of the series. Furthermore since the sign

* While it is necessary to distinguish between the two kinds of infinite determinants, it would seem that there is no loss of generality in confining our investigations to the determinants of the second kind; for in the definition of an infinite determinant m is simply supposed to become indefinitely great, and making M infinite secures this; however, adopting the safer plan, I shall confine my investigations to determinants of the second kind.

and the order of the factors constituting any term are determined by the rules already given, the only effect introduced by making a change in the choice of principal diagonal or origin, is a change in the order of the terms of the series; hence *the value of an absolutely convergent determinant is independent of the choice of principal diagonal or origin.*

Since a semi-convergent determinant represents a semi-convergent series, the value of such a determinant is dependent upon the choice of the law which will fix the order of the terms of the series. If we adopt any rule to determine the order of the terms, we may investigate the conditions of convergence corresponding to that particular rule; but as the number of such rules from which a choice may be made is infinite, it would be useless to investigate the conditions for any one rule, unless it can be shown that for sufficient reasons that rule is to be preferred to all others. It would seem that this is not the case, so that it is not probable that much can ever be done with this class of determinants.

Since a change in the origin must introduce a change in the order of the terms of the series, it follows that *a semi-convergent determinant is dependent upon the choice of its origin for its value.*

6. Theorems on convergent determinants.

Following the method generally adopted in the treatment of infinite series, we shall prove certain theorems concerning convergent determinants, reserving for the present the investigation of the question of convergence. Considerable generality can be given to such theorems owing to the fact that their proof rests upon the proposition that a convergent infinite determinant can be represented by a convergent infinite series, the terms of which are arranged according to some definite law; a little reflection will show that the proof is independent of the particular law that might be chosen.

Theorem I. An absolutely convergent determinant remains absolutely convergent if we replace the elements of any line (or column) by a series of quantities all of which are less in absolute value than a given positive quantity.

Let us represent the infinite determinant D_∞ by the expression

$$\sum_{-\infty}^{+\infty}{}_i B_i\,, \tag{1}$$

where B_i represents the product of an infinite number of factors A_{ik}, the sign of any term and the order of the factors in that term being governed by the rules already adopted, and the order of the terms of the series being fixed by some law which we shall suppose known.

To fix the idea let us suppose the elements of the line 0

$$\ldots, A_{0,-m}, \ldots, A_{00}, \ldots, A_{0m}, \ldots$$

to be replaced by the quantities

$$\ldots, s_{-m}, \ldots, s_0, \ldots, s_m, \ldots$$

satisfying the inequalities

$$|s_i| < k. \qquad k > 0.$$

By hypothesis the series

$$\sum_{-\infty}^{+\infty}{}_i |B_i|$$

is convergent ; consequently the series

$$\sum_{-\infty}^{+\infty}{}_i k |B_i'|, \tag{2}$$

where B_i' represents B_i divided by its factor from the line 0, is convergent ; the series

$$\sum_{-\infty}^{+\infty}{}_i |C_i|, \tag{3}$$

where C_i represents B_i' multiplied by the s corresponding to its element from the line 0, is also convergent since its terms are respectively equal to or less than the corresponding terms of the series (2). Since the series (3) is convergent, the determinant which represents

$$\sum_{-\infty}^{+\infty}{}_i C_i$$

is absolutely convergent, and the theorem is proved.

Cor. 1. An absolutely convergent determinant remains absolutely convergent if we replace any finite number of lines and columns by series of quantities all of which are less in absolute value than a given positive quantity.

This proposition regarding infinite determinants is the analogue of the proposition that in considering the convergence of a series we may neglect any finite number of terms. This theorem was first proved by Poincaré for the class of determinants which he investigated, and was afterwards extended by von Koch to determinants of the normal form.

Cor. 2. A semi-convergent determinant remains semi-convergent if we replace the elements of any finite number of lines and columns by series of quantities all of which are less than a given positive quantity, and each of which has the same sign as the element that it replaces.

The proof of this proposition is the same as that of the theorem for absolutely convergent determinants; the necessity for making the restriction that the quantities have the same sign as the elements they replace is made clear by considering the case in which the terms of the series representing the semi-convergent determinant are some negative and the others positive; the signs of the s's might be such as to make the terms of the series represented by the new determinant all positive; since the original determinant is only semi-convergent, the series thus formed is divergent, and consequently the determinant has lost its convergence by replacing its elements by the s's. If however the s's all have the same signs as the elements they replace, the signs of the terms of the series cannot be changed, and the determinant remains semi-convergent when we replace the elements of a given line by the s's.

Cor. 3. A determinant (of the second kind) which is of the normal form is an absolutely convergent determinant.

Every convergent determinant of the second kind must belong to one of the two classes, semi-convergent or absolutely convergent determinants. Von Koch has proved that a determinant of the normal form is convergent, and that if in such a determinant we replace the elements of any line or column by any series of quantities, positive or negative, all of which are less in absolute value than a given positive quantity, the determinant does not lose its convergence; it follows from the preceding theorem that this is the case *without restriction* only for absolutely convergent determinants; hence determinants of the normal form (when of the second kind) come under the class of absolutely convergent determinants, and any proposition proved for the latter is also true for the former.

Theorem II. If in an absolutely convergent determinant we interchange any two lines (or columns), the sign of the determinant is changed; if any two lines (or columns) are identical the determinant vanishes.

Since the determinant is convergent, we have

$$|D_{m+p} - D_m| < \delta. \tag{1}$$

If we denote by D'_m, the value of the determinant D_m when the two lines (or columns) are interchanged, we have by the preceding theorem

$$|D'_{m+p} - D'_m| < \delta. \tag{2}$$

Now m can always be chosen so large that the two lines (or columns) interchanged will be included in D_m; hence

$$D'_m = - D_m. \tag{3}$$

But we have

$$|D_{m+p} - D_m + D'_{m+p} - D'_m| \leqq 2\delta, \tag{4}$$

or in consequence of (3)

$$|D_{m+p} - D'_{m+p}| \leqq 2\delta; \tag{5}$$

hence if we represent by D and D' the limits towards which D_m and D'_m tend as m increases indefinitely, we have for m infinitely great

$$D = - D'. \tag{6}$$

The second part of the theorem follows immediately; if two lines (or columns) are identical, the value of the determinant can not be altered by interchanging them; but the sign of the determinant must be altered by this interchange, hence the determinant vanishes.

Cor. If a semi-convergent determinant does not lose its convergence when we interchange two lines (or columns) the sign of the determinant is changed; if any two lines (or columns) are identical, the determinant vanishes.

The proof of this proposition is the same as that for absolutely convergent determinants.

Theorem III. If every element of any line (or column) of a convergent determinant be multiplied by the same factor k, the determinant is multiplied by that factor.

Since every term of the series

$$\sum_{-\infty}^{+\infty}{}_i B_i \tag{1}$$

is a convergent product, and contains one and only one element from any line (or column), each term of the series is multiplied by the factor k; and since the series is convergent, the series formed by multiplying each term by the same factor is convergent, and has for its value

$$k \sum_{-\infty}^{+\infty}{}_i B_i; \tag{2}$$

hence if the value of the original determinant is D, the value of the new determinant is kD.

Cor. 1. If the elements of any line (or column) are the same multiples of the corresponding elements of some other line (or column) the determinant vanishes.

By the theorem just given we can divide the elements of that line (or column) by the common factor; two lines (or columns) being then identical, the determinant vanishes.

Cor. 2. If the signs of all the elements of any line (or column) be changed, the sign of the determinant is changed.

Such a change of sign of the elements is equivalent to multiplying these elements, and consequently the determinant, by the factor -1.

Theorem IV. If each element of any line (or column) of an absolutely convergent determinant can be resolved into the sum of n quantities, the given determinant can be expressed as the sum of n absolutely convergent determinants.

To fix the idea, let us suppose that the elements of the line 0 can be resolved into the sum of n quantities, so that we have

$$A_{0m} = a_{1m} + a_{2m} + a_{3m} + \ldots + a_{nm}\,.$$

Owing to the absolute convergence of the series

$$\sum_{-\infty}^{+\infty}{}_i\, B_i\,,$$

each of the series

$$\sum_{-\infty}^{+\infty}{}_i\, a_{ri}B'_i \qquad (r = 1, 2, \ldots, n.)$$

is absolutely convergent; hence we can write

$$\sum_{-\infty}^{+\infty}{}_i\, B_i = \sum_{-\infty}^{+\infty}{}_i\,(a_{1i} + a_{2i} + \ldots + a_{ni})\, B'_i$$

$$= \sum_{-\infty}^{+\infty}{}_i\, a_{1i}B'_i + \sum_{-\infty}^{+\infty}{}_i\, a_{2i}B'_i + \ldots + \sum_{-\infty}^{+\infty}{}_i\, a_{ni}B'_i\,;$$

or

$$D = \Delta_1 + \Delta_2 + \ldots + \Delta_n\,,$$

where D is the original determinant, and Δ_r is the determinant formed by replacing the elements of the line 0 of the original determinant by the quantities $a_{ri}\,(i = -\infty, \ldots, +\infty)$. By Theorem I each of these determinants is absolutely convergent, and the theorem under consideration is thus proved.

Cor. The theorem holds for a semi-convergent determinant, if all the elements of the line 0 in each of the determinants Δ_r have the same sign as the corresponding elements in the determinant D.

This follows as a consequence of Theorem I, Cor. 2.

7. Tests of convergence.

Before attempting a general discussion of the conditions of convergence for infinite determinants, we shall give two theorems concerning determinants of special form, which may be interesting in themselves, as well as of use subsequently.

THEOREM I. An infinite determinant all the terms of which are negative is divergent.

Any term of the series

$$\sum_{-\infty}^{+\infty}{}_i B_i$$

consists of the product of an infinite number of negative factors; the product of an even number of these factors gives a positive quantity; the introduction of another factor changes the sign of the product, and as we thus increase the number of the factors, the sign of the product continually changes from plus to minus; the product is therefore divergent, and consequently the determinant is divergent.

THEOREM II. In order that an infinite determinant, all the terms of which are positive, may be convergent, it is necessary and sufficient that the terms of the series

$$\sum_{-\infty}^{+\infty}{}_i B_i$$

tend to zero as a limit.

Since the elements of the determinant are all positive, it will follow as a consequence of the law governing the sign of any term, that half of the terms of the series

$$\sum_{-\infty}^{+\infty}{}_i B_i$$

are positive and the other half negative; in whatever manner they may be grouped only a finite number of positive or negative terms can fall together; since these terms by hypothesis tend to zero as a limit, the sum of any finite number of terms will also tend to zero as a limit; the series has thus been reduced to one in which the terms are alternately positive and negative, and tend to zero as a limit; such a series is known to be convergent, and the theorem is proved.

The consideration of these two theorems places us in a better position for formulating the necessary conditions that an infinite determinant be convergent. In the first place in order that an infinite determinant which is represented by the series

$$\sum_{-\infty}^{+\infty}{}_i B_i$$

be convergent, it is necessary that each one of the infinite products B_i be convergent. Such an infinite product may become divergent in one of two ways; it may either tend to infinity as a limit, or its sign may be alternately positive and negative owing to the repeated introduction of a negative factor.

Let us write $A_{ik} = \pm (1 + a_{ik})$; if any of the quantities A_{ik} are less in absolute value than unity, the corresponding a_{ik} is to be taken as equal to zero. Let us now form the series

$$\Sigma a_{ik} ,$$

where the quantities a_{ik} are selected and arranged according to the same rules as the quantities A_{ik} in the products B_i; there will be an infinite number of these series, one corresponding to each product B_i; let us represent these series by S_i ($i = -\infty , \ldots , + \infty$). In order that any product B_i be convergent it is necessary that the corresponding series S_i shall be convergent; hence we are able to state as a first necessary condition that the infinite determinant be convergent, that each one of the series S_i ($i = -\infty , \ldots , + \infty$) must be convergent.

Any one of the terms B_i will be divergent if it contains an infinite number of negative factors which are in absolute value equal to or greater than unity. An infinite number of negative factors all less in absolute value than unity would not make it divergent, for while its sign would be alternately positive and negative, it would tend to zero as a limit; hence we may state as a second necessary condition that any term B_i must contain only a finite number of negative factors satisfying the inequality

$$| A_{ik} | \geqq 1 .$$

Supposing that the convergence of each term of the series

$$\sum_{-\infty}^{+\infty}{}_i \, B_i$$

is secured by the fulfilling of the two conditions already given, in order that the infinite series may be convergent it is necessary that its terms tend to zero as a limit; that is, we must always be able to find an integer n', such that for any value of n greater than n'

$$| B_{\pm n} | < \delta ,$$

δ being any positive quantity which may be chosen as small as we please; in order that this inequality may exist, each term $B_{\pm n}$ ($n > n'$) must either contain a factor which is evanescent, or it must contain an infinite number of factors each less than unity.

To give a résumé of the results arrived at, we may state that in order that an infinite determinant may be convergent, the following conditions are necessary:—

1. Each of the series S_i ($i = -\infty , \ldots , + \infty$) must be convergent.

2. Any term must contain only a finite number of negative factors satisfying the inequality

$$|A_{ik}| \geqq 1 .$$

3. Each term after a certain term in the series is reached must either contain an evanescent factor or an infinite number of factors each of which is less than unity.

While these are necessary conditions, they are not in general sufficient to determine the convergence of a determinant; however, in connection with the preceding theorems they enable us to state that certain classes of determinants are convergent; for example, any determinant satisfying the following conditions is convergent: (a) the elements are all positive; (b) each of the series $S_i \, (i = -\infty, \ldots, +\infty)$ is convergent; (c) for some line i (or column k) the elements A_{ik} tend to zero as we go farther from the origin.

Also any determinant satisfying the following conditions is convergent: (a) the elements are all positive; (b) each of the series $S_i \, (i = -\infty, \ldots, +\infty)$ is convergent; (c) an infinitely great number of lines (or columns) are composed of quantities less in value than unity.

Nov. 5, 1894.

ON THE CALCULUS OF FUNCTIONS DERIVED FROM LIMITING-RATIOS.*

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I. Introduction.

1. The object of this essay is an investigation of the foundation of the Calculus of Limiting-Ratios as a particular case of the Calculus of Finite Differences considered in a generalized sense. It possesses the scientific interest of showing that the Differential Calculus of Newton is but one particular case of an infinite number of such Calculi, and is the simplest of them all.

The limiting-ratios of the Calculus as founded by Newton and Leibnitz, or as they are usually called the successive derivatives of $f(x)$, are defined to be

$$f^{m+1}(x) = \mathop{\mathcal{L}}_{h=0} \frac{f^m(x+h) - f^m(x)}{h},$$

$m = 0, 1, 2, \ldots$, giving the successive derivatives. These functions are derived, each from the preceding, through exactly the same operation, which is therefore essentially a repetitive operation. The Calculus of Derivatives is a repetitive calculus whose operator is (d/dx), and the repetitive character is symbolized by the operative index following the exponential law

$$\left[\frac{d}{dx}\right]\left[\frac{d^n}{dx^n}\right] = \left[\frac{d^{n+1}}{dx^{n+1}}\right].$$

In like manner, the Calculus of Finite Differences, as ordinarily considered, is also a repetitive calculus. The successive functions of this Calculus are defined to be

$$\Delta_h^m f(x),$$

wherein, if

$$E_h^m f(x) = f(x + mh),$$

then

$$\Delta_h^m f(x) = (E_h - 1)^m f(x),$$

using the notations usually employed in the Calculi of Finite Differences and Enlargement.

The Differential Calculus can be otherwise founded as a particular case

* Read before The American Mathematical Society in New York, May, 1895.

of the Calculus of Finite Differences, and the successive derivatives defined to be the functions

$$f^{m}(x) = \mathop{\mathcal{L}}_{\Delta x=0} \frac{\Delta_h^m f(x)}{(\Delta x)^m} = \mathop{\mathcal{L}}_{h=0} \frac{\Delta_h^m f(x)}{h^m}. \qquad (m = 1, 2, 3, \ldots)$$

The repetitive operator (Δ_h^m/h^m) furnishing what is called the successive difference-ratios of the function, and

$$\frac{\Delta}{h}\left[\frac{\Delta^n}{h^n}\right] = \left[\frac{\Delta^{n+1}}{h^{n+1}}\right].$$

2. It is to be observed, in the Calculus of Finite Differences, that the interval between the arguments is constantly equal to h, whence all successive differences of these arguments vanish after the first. It is now proposed to generalize the Calculus of Finite Differences and to naturalize the Calculi of Limiting-Ratios as particular cases of this Calculus. Throughout this paper, $f(x)$, unless otherwise specifically stated, will be taken to represent a uniform, finite, and continuous function of x which is expressible by Taylor's series throughout a certain definite interval, say, from $x = a$ to $x = c$.

We define ε to be an operator whose function it is to act on the variable as follows:

$$\varepsilon_h^r \cdot f(x_g) = f(x_{g+rh}).$$

We define δ to be an operator which produces the effect

$$\begin{aligned} \delta_h^n \cdot f(x_g) &= (1 - \varepsilon_h)^n \cdot f(x_g), \\ &= f(x_g) - C_{n,1} f(x_{g+h}) + C_{n,2} f(x_{g+2h}) - \ldots + (-1)^n f(x_{g+nh}). \end{aligned}$$

Thus the operations have the relations

$$\delta_h = 1 - \varepsilon_h, \quad \varepsilon_h = 1 - \delta_h.$$

The corresponding nth difference of the argument is

$$\begin{aligned} \delta_h^n \cdot x_g &= (1 - \varepsilon_h)^n \cdot x_g \\ &= x_g - C_{n,1} x_{g+h} + C_{n,2} x_{g+2h} - \ldots + (-1)^n x_{g+nh}. \end{aligned}$$

We shall be concerned with the limiting value of the ratio

$$\mathop{\mathcal{L}}_{h=0} \frac{\delta_h^n f(x_g)}{\delta_h^n x_g},$$

which is the nth generalized-difference-ratio of the nth generalized difference of the function to a corresponding nth difference of the variable. We call the

above function the nth difference-ratio-limit, and at the risk of being guilty of surreptitious coinage, we abbreviate this into nth *differell.*

A symbolic notation for the operation will be required, and we write the nth differell of $f(x)$

$$\left[\frac{d^n}{d^n x}\right] f(x) = f^{(n)}(x) = \mathop{\pounds}_{h=0} \frac{\delta_h^n f(x)}{\delta_h^n x},$$

as distinguished from the notations of the Differential Calculus,

$$\left[\frac{d^n}{dx^n}\right] f(x) = f^n(x) = \mathop{\pounds}_{h=0} \frac{\Delta_h^n f(x)}{h^n}.$$

3. The law of the distribution of the argument x_{g+rh} is perfectly arbitrary, except that x_{g+rh} must converge to the limit x_g as h converges to zero. In general, for an assigned law of argument distribution we shall have a Differell Calculus in which the differell operator is not repetitive, and for a different argument distribution law we shall have a different Differell Calculus. In each of these calculi any differell operator may be taken as an operator *per se*, as the fundamental operation of a new calculus in which the operations are repetitive. The first differell in any such Differell Calculus is the derivative of the function. For

$$\mathop{\pounds}_{h=0} \frac{\delta' f(x_g)}{\delta' x_g} = \mathop{\pounds}_{h=0} \frac{f(x_g) - f(x_{g+h})}{x_g - x_{g+h}} = f'(x_g).$$

4. Two general distribution laws suggest themselves at once,

$$x_{g+rh} = x_g + \varphi(r, h),$$

$$x_{g+rh} = x_g\, \psi(r, h).$$

In which $\varphi(r, h)_{h=0} = 0$ and $\psi(r, h)_{h=0} = 1$.

We may call the former the Addition Calculus and the latter the Multiplication Calculus. This latter term has already been used by Dr. McClintock in his Calculus of Enlargement,* wherein, at the close, he makes mention of such a calculus.

Of the Addition Calculi, the simplest is given by the law

$$\varphi(r, h) = h.$$

All differells, after the first are infinite. The first exists, however, and taken as an operation *per se*, may be made the basis of a repetitive calculus, which

* American Journal of Mathematics, Vol. II, p. 156.

is the Differential Calculus. Such is the case for the first differell of any Differell Calculus. As another illustration, let

$$x_{g+rh} = x_g - C_{r,1}h + C_{r,2}h^2 - \ldots + (-1)^r h^r = x_g - 1 + (1-h)^r .$$

Here, evidently,

$$\delta_h^n x_g = h^n .$$

And it appears that the nth differell of this calculus is

$$f^{(n)}(x) = f'(x) + f''(x) + \ldots + f^n(x) .$$

Any rth differell may be taken as the fundamental operation of a repetitive calculus.

In like manner the simplest Multiplication Calculus is furnished by the law

$$x_{g+rh} = x_g(1-h)^r ,$$

or what is the same thing, for the results are the same,

$$x_{g+rh} = x_g e^{rh} .$$

Because this form of the calculus has been considered symbolically by Dr. McClintock, in the paper above referred to, we shall proceed to notice it particularly as an illustration of what may be developed in other calculi.

II. The Multiplication Calculus.—Differells

5. Let the law of distribution be

$$x_{g+rh} = x_g(1-h)^r$$

or as it may be written

$$\log x_{g+rh} - \log x_g = r \log(1-h) .$$

The interval between the logarithms of the arguments is $\log(1-h)$. Otherwise, if $x_{g+rh} = x_g e^{rh}$, we have

$$\log x_{g+rh} - \log x_g = rh ,$$

and the interval between the logarithms of the argument is simply h

In the first case we have

$$\delta_h^n x_g = x_g h^n ,$$

$$\delta_h^n x_g^m = x_g^m [1 - (1-h)^m]^n .$$

On taking the ratio and passing to the limit, we obtain

$$\left[\frac{d^n}{d^n x}\right] x^m = m^n x^{m-1}.$$

In the second case, we have

$$\delta_h^n x_g = x_g (1 - e^h)^n,$$

$$\delta_h^n x_g^m = x_g^m (1 - e^{mh})^n,$$

and as before

$$\left[\frac{d^n}{d^n x}\right] x^m = m^n x^{m-1}.$$

6. From the nature of the formation of the nth generalized difference, we have

$$\left[\frac{\delta_h^n}{\delta_h^n x}\right] [\varphi(x) + \psi(x)] = \frac{\delta_h^n \varphi(x)}{\delta_h^n x} + \frac{\delta_h^n \psi(x)}{\delta_h^n x},$$

and on passing to the limit,

$$\left[\frac{d^n}{d^n x}\right] [\varphi(x) + \psi(x)] = \varphi^{(n)}(x) + \psi^{(n)}(x).$$

Or, the differell of a sum of functions is the sum of the differells of the functions.

7. Let $nh = k$, a constant, so when $h = 0$, $n = \infty$. Then, if $0 < q < 1$, we have at any point x_{g+qk} between x_g and x_{g+k},

$$x_{g+qk} - x_{g+qk+rh} = x_g e^{qk} (1 - e^{rh}),$$

and by taking h small enough we have the distribution of the argument as nearly uniform as we choose throughout the neighborhood of any point x_{g+qk} in the interval (x_g, x_{g+k}).

8. To examine the formation of the differells of functions, we have by Taylor's formula,

$$f(a + x) = \sum_0^\infty \frac{x^r}{r!} f^r(a).$$

The nth differell of this function with respect to x, is

$$f^{(n)}(a + x) = \sum_1^\infty r^n \frac{x^{r-1}}{r!} f^r(a),$$

a series which is convergent when Taylor's is, since they have the same ratio of convergency.

For $a + x$ substitute x, then

$$f^{(n)}(x) = \sum_1^{\infty} r^n \frac{(x-a)^{r-1}}{r!} f^r(a).$$

The next differell is

$$f^{(n+1)}(x) = \sum_1^{\infty} r^{n+1} \frac{(x-a)^{r-1}}{r!} f^r(a).$$

Multiply $f^{(n)}(x)$ by $(x - a)$ and differentiate, whence

$$f^{(n+1)}(x) = \frac{d}{dx}\left[(x-a) f^{(n)}(x)\right].$$

Again, multiply $f^{(n+1)}(x)$ by dx and integrate between x and a, whence results,

$$\int_a^x f^{(n+1)}(x)\, dx = (x-a) f^{(n)}(x).$$

Thus we find that $f^{(n)}(x)$ is the mean value of $f^{(n+1)}(x)$ between x and a. $f^{(n)}(x)$ is therefore a function of an interval, and has distinct reference to an argument interval, one end of which is a, called the base of reference of differells.

When the interval vanishes, we have, in general,

$$f^{(n+1)}(a) = f^{(n)}(a) = f'(a).$$

If zero is the base, we have correspondingly

$$f^{(n+1)}(x) = \frac{d}{dx}\left[x f^{(n)}(x)\right],$$

$$\int_0^x f^{(n+1)}(x)\, dx = x f^{(n)}(x).$$

9. We shall use the base a, unless specifically mentioned. By successive applications of the formulæ of the preceding article, we may express $f^{(n)}(x)$ in terms of the first n derivatives of $f(x)$.

We have, by definition,

$$f^{(1)}(x) = f'(x).$$

$$\begin{aligned}
\therefore\ f^{(2)}(x) &= f'(x) + (x-a) f''(x),\\
f^{(3)}(x) &= f'(x) + 3(x-a) f''(x) + (x-a)^2 f'''(x),\\
f^{(4)}(x) &= f'(x) + 7(x-a) f''(x) + 6(x-a)^2 f'''(x) + (x-a)^3 f^{iv}(x),\\
f^{(5)}(x) &= f'(x) + 15(x-a) f''(x) + 25(x-a)^2 f'''(x) + 10(x-a)^3 f^{iv}(x)\\
&\quad + (x-a)^4 f^{v}(x),
\end{aligned}$$

and so on.

Putting

$$f^{(n)}(x) = \sum_1^n A_r(x-a)^{r-1}f^r(x)\,,$$

$$f^{(n+1)}(x) = \sum_1^{n+1} A_r'(x-a)^{r-1}f^r(x)\,,$$

we have for the law of formation

$$A_r' = rA_r + A_{r-1}\,,$$

and

$$A_1 = 1\,,\quad A_0 = A_{n+1} = 0\,.$$

We may express this law more concisely in several ways, thus :*

$$(x-a)f^{(n)}(x) = \left[\frac{d}{dy}\right]^n f(x) = \left[(x-a)\frac{d}{dx}\right]^n f(x)$$

$$= \sum_1^n \Delta^r 0^n \,.\, \frac{(x-a)^r}{r!} f^r(x)\,,$$

wherein $x - a = e^y$, and $\Delta^r 0^n$ is the well known number in the Finite Difference Calculus.

Also

$$f^{(n)}(x) = \left[1 + \frac{d}{dy}\right]^{n-1} f'(x)\,.$$

Again,

$$f^{(n+1)}(x) = \left[\frac{d}{dx}(x-a)\right]^n f'(x)\,,$$

$$f^{(n+r)}(x) = \left[\frac{d}{dx}(x-a)\right]^r f^{(n)}(x)\,,$$

in which the exponent of the square bracket indicates successive applications of the enclosed operator.

10. If we take the expressions of the above article as defining differells, we show that

$$\left[\frac{d^n}{d^n x}\right](x-a)^m = m^n(x-a)^{m-1},$$

where a is the base, as follows :—

$$f^{(2)}(x) = \frac{d}{dx}[(x-a)f'(x)] = \frac{d}{dx}[m(x-a)^m] = m^2(x-a)^{m-1},$$

$$f^{(3)}(x) = \frac{d}{dx}[(x-a)\,m^2(x-a)^{m-1}] = m^3(x-a)^{m-1}.$$

* Herschel's Theorem. See Laurent, Traité d'Analyse, Vol. III, p. 436.

If it is true for p, then

$$f^{(p+1)}(x) = \frac{d}{dx}[(x-a)\,m^p\,(x-a)^{m-1}] = m^{p+1}(x-a)^{m-1},$$

which establishes the result generally.

11. We can, by reversing the process of Art. 9, express $f^{(n)}(x)$ in terms of the first n differells, thus :—

$$\begin{aligned}(x-a)\,f''(x) &= f^{(2)}(x) - f^{(1)}(x)\,,\\ (x-a)^2 f'''(x) &= f^{(3)}(x) - 3f^{(2)}(x) + 2f^{(1)}(x)\,,\\ (x-a)^3 f^{\text{iv}}(x) &= f^{(4)}(x) - 6f^{(3)}(x) + 11f^{(2)}(x) - 6f^{(1)}(x)\,,\end{aligned}$$

and so on. Or, generally,

$$(-1)^n\,(x-a)^{n-1} f^n(x) = \sum_1^n A_r f^{(r)}(x)\,,$$

$$(-1)^{n+1}\,(x-a)^n\,f^{n+1}(x) = \sum_1^{n+1} A_r' f^{(r)}(x)\,,$$

wherein

$$A_r' = nA_r - A_{r-1}\,,$$

and

$$A_n = 1\,,\quad A_0 = A_{n+1} = 0\,.$$

Since, when $x = a$, $f^{(r)}(a) = f'(a)$, we must have $\sum_1^n A_r = 0$. The above results serve to define the nth derivative as

$$f^n(a) = \underset{x=a}{\pounds}\,\frac{\sum^n A_r f^{(r)}(x)}{(-1)^n\,(x-a)^{n-1}}\,.$$

The general law for the expression of the nth derivative in terms of the first n differells will be given in compact form later.

12. We have heretofore considered h as a positive convergent to zero; when this is so and $0 < h < 1$, the arguments are distributed, we shall say progressively, from x_g in the order $x_{g+rh}, \ldots, x_{g+h}, x_g$ for the binomial form of distribution $x_g\,(1-h)^r$, and in the order $x_g, x_{g+h}, \ldots, x_{g+rh}$ for the exponential form $x_g e^{rh}$. If h is a negative convergent to zero the distributions are regressive and both in the binomial and in the exponential form are in the order $x_{g+rh}, \ldots, x_{g+h}, x_g$. It is easily seen that the corresponding progressive and regressive differells are the same functions in either case.

13. *Fractional Differells.* Heretofore we have considered the order of a differell as an integer, but as these operations are not repetitive, there is nothing in the formation of these functions to thus restrict us.

Let ρ be any numerical ratio, then

$$\varepsilon_h^\rho x_g = x_{g+\rho h} ,$$

$$\begin{aligned} \delta_h^\rho x_g &= (1 - \varepsilon_h)^\rho x_g , \\ &= (1 - C_{\rho,1}\,\varepsilon_h + C_{\rho,2}\,\varepsilon_h^2 - \ldots)\, x_g . \end{aligned}$$

The expression of an operator as an infinite series is one so frequently made use of, that we shall use it here in the same sense as it is employed elsewhere in mathematical investigations.

$$\therefore \qquad \delta_h^\rho x_g = x_g - C_{\rho,1}\, x_{g+h} + C_{\rho,2}\, x_{g+2h} - \ldots .$$

This series must be convergent in order to give meaning to the results. If the law of distribution is $x_g(1 - h)^r$, and h is a positive number between zero and two, the series does converge, and we have

$$\begin{aligned} \delta_h^\rho x_g &= x_g\,[1 - (1 - h)]^\rho , \\ &= x_g h^\rho . \end{aligned}$$

In like manner, we have

$$\delta_h^\rho x_g^m = x_g^m\,[1 - (1 - h)^m]^\rho ,$$

and there results from the limiting ratio,*

$$\left[\frac{d^\rho}{d^\rho x}\right] x^m = m^\rho x^{m-1} ,$$

the same function as when ρ is an integer. When h is a negative convergent to zero, the infinite series above diverges, and we cannot say whether the regressive fractional derivative exists or not.

If, however, h is a negative convergent to zero and the law is $x_g e^{\rho h}$, the series converges and gives the same result as above, while in this case when h is a positive convergent the series is divergent.

In order to derive all the justification we can for these results, consider the law of distribution to be $x_g e^{\rho h}$. Now, n being integer, $\varepsilon_h^n x_g^m = x_{g+nh}^m$ for any value of m; and

$$\delta_h^n x_g^m = (1 - \varepsilon_h)^n x_g^m .$$

But $\varepsilon_h^n x_g^m = (e^{mh})^n x_g^m$. Therefore, so far as the operation on x_g^m is concerned we may replace ε_h by e^{mh}. We have

$$\frac{\delta_h^\rho x_g^m}{\delta_h^\rho x_g} = \frac{(1 - \varepsilon_h)^\rho x_g^m}{(1 - \varepsilon_h)^\rho x_g} = \frac{(1 - e^{mh})^\rho x_g^m}{(1 - e^h)\, x_g} .$$

* The base of operation here is zero.

Whether h be positive or negative, we have

$$\left[\frac{d^\rho}{d^\rho x}\right] x^m = m^\rho x^{m-1}.$$

Another symbolical verification of this result follows; we have, in § 9,

$$f^{(\rho)}(x) = \left[1 + \frac{d}{dy}\right]^{\rho-1} f_x'(e^y),$$

when ρ is an integer; assuming it true when ρ is fractional, let $f(x) = x^q$, then $f'(x) = qx^{q-1} = qe^{(q-1)y}$, and

$$f^{(\rho)}(x) = (1 + C_{\rho-1,1} D_y + C_{\rho-1,2} D_y^2 + \ldots) qe^{(q-1)y}$$
$$= qe^{(q-1)y} [1 + C_{\rho-1,1}(q-1) + C_{\rho-1,2}(q-1)^2 + \ldots].$$

If $-1 < q - 1 < +1$, the series converges, and

$$f^{(\rho)}(x) = q^\rho e^{(q-1)y} = q^\rho x^{q-1}.$$

For integral values of ρ we have the Differell Calculus as developed in the preceding articles. For each value of ρ fractional or integral and positive, we can develop a calculus in which the operative symbol is

$$\left[\frac{d^\rho}{d^\rho x}\right],$$

which may be taken as a repetitive operation.

Thus

$$\left[\frac{d^\rho}{d^\rho x}\right]\left[\frac{d^\rho}{d^\rho x}\right] = \left[\frac{d^\rho}{d^\rho x}\right]^2,$$

and so on. In any such Rho-Differell Calculus, we have

$$\left[\frac{d^\rho}{d^\rho x}\right]^m x^q = q^\rho (q-1)^\rho \ldots (q-m+1)^\rho x^{q-m}$$

which under certain conditions may be imaginary. This degenerates into the Differential Calculus when $\rho = 1$.

14. *Fractional Derivatives.* The close relations existing between the Differell Calculus and the Differential Calculus, and the existence of the functions of the former when the operational index is fractional suggests an effort to define the operation upon the functions of the latter when the index of that operation is a fraction, through the existing relations of the operators in the two calculi. At the outset it must be seen that no fractional index of an

operation can be the result of a repetitive operation, but must indicate an operation *per se*, and upon the basis of this necessary truth rests the convention upon which depends the following interpretation.

Letting the base of differells be zero for convenience, we have, when n is a positive integer,

$$xf^{(n)}(x) = \left[\frac{d}{dy}\right]^n f(e^y),$$

wherein $x = e^y$. Let $f(x) = x^q$ (q being any number), then $f(e^y) = e^{qy}$, and we have from the Differell Calculus

$$xf^{(n)}(x) = xq^n x^{q-1} = q^n x^q.$$

From the Differential Calculus

$$\frac{d^n}{dy^n} e^{qy} = q^n e^{qy} = q^n x^q.$$

But when n is any positive fraction, say ρ, we have still

$$xf^{(\rho)}(x) = x\left[\frac{d^\rho}{d^\rho x}\right] x^q = q^\rho x^q.$$

And in the equalities

$$\left[\frac{d^\rho}{dy^\rho}\right] x^q = x\left[\frac{d^\rho}{d\rho x}\right] x^q = q^\rho x^q,$$

we know that both equalities hold good for ρ a positive integer, and that the second equality is true for ρ any positive number. Let us assume that the first equality is also true when ρ is a positive fraction, thus defining the fractional operator in the symbol (d^ρ/dy^ρ). When ρ is an integer, the ordinary signification of

$$\left[\frac{d^\rho}{dy^\rho}\right] = \left[\frac{d^\rho}{(dy)^\rho}\right] = \left[\frac{d}{dy}\right]^\rho,$$

means that the operator d/dy is to be successively performed on a function ρ times, but when ρ is fractional no such repetitive meaning is possible for this operation and we can only interpret it as an operation *per se*.

The question arises, if

$$\left[\frac{d^\rho}{dy^\rho}\right] e^{qy} = q^\rho e^{qy},$$

is to be true for all positive values of ρ, what is the value of

$$\left[\frac{d^\rho}{dy^\rho}\right] y^q.$$

It is necessary that ρ should be a non-repetitive operator, whether integral or fractional, and considered as the symbol of an operation in itself. We have $x = e^y$ and $dy = dx/x$. Therefore

$$\left[\frac{d^\rho}{dy^\rho}\right] e^{qy} = \left[\frac{d^\rho}{(dx/x)^\rho}\right] x^q ,$$

which we write

$$\left[x^\rho \frac{d^\rho}{dx^\rho}\right] x^q ,$$

the value of which, is by the above definition $q^\rho e^{qy} = q^\rho x^q$.

When $\rho = 1$, we have

$$\left[x \frac{d}{dx}\right] = x \left[\frac{d}{dx}\right].$$

We *assume*, inasmuch as ρ is no longer indicative of a repetitive operation, that

$$\left[x \frac{d}{dx}\right]^\rho = \left[x^\rho \frac{d^\rho}{dx^\rho}\right] = x^\rho \left[\frac{d^\rho}{dx^\rho}\right].$$

Under this convention, we have

$$\left[\frac{d^\rho}{dx^\rho}\right] x^q = q^\rho \, x^{q-\rho} .$$

Otherwise, using the separation of symbols, as in the infinitesimal calculus, we have

$$\begin{aligned} d^\rho x^q &= q^\rho x^q \,[d\,(\log x)]^\rho \\ &= q^\rho x^q \left[\frac{dx}{x}\right]^\rho \\ &= q^\rho x^{q-\rho} (dx)^\rho . \end{aligned}$$

For any positive value of ρ, we can develop a calculus in which the fundamental operative symbol is

$$\left[\frac{d^\rho}{dx^\rho}\right],$$

which can be taken repeatedly furnishing the repetitive operation of this particular calculus. Thus

$$\left[\frac{d^\rho}{dx^\rho}\right] \left[\frac{d^\rho}{dx^\rho}\right] = \left[\frac{d^\rho}{dx^\rho}\right]^2$$

and so on. But the exponential law

$$\left[\frac{d^\rho}{dx^\rho}\right]^2 = \frac{d^{\rho+2}}{dx^{\rho+2}}$$

does not hold. In any such Rho-Differential Calculus, we have, assuming m to be an integer,

$$\left[\frac{d^\rho}{dx^\rho}\right]^m x^q = q^\rho (q-\rho)^\rho \ldots (q-\overline{m-1}\rho)^\rho x^{q-m\rho},$$

which degenerates into the ordinary Differential Calculus when $\rho = 1$.

III. Expansion of Functions.

15. If the differell of a function vanishes for two values of the variable, the differell of next higher order must vanish for some value of the variable which lies between these two.

In virtue of $f^{(n)}(x)$ being the mean value of $f^{(n+1)}(x)$ between a and x, we have

$$f^{(n)}(x) = f^{(n+1)}(u). \qquad a < u < x.$$

Now

$$\int_b^c f^{(n+1)}(x)\,dx = \int_a^c f^{(n+1)}(x)\,dx - \int_a^b f^{(n+1)}(x)\,dx$$

$$= (c-a) f^{(n)}(c) - (b-a) f^{(n)}(b)$$

$$= (c-b) f^{(n+1)}(u). \qquad b < u < c.$$

If $f^{(n)}(c) = f^{(n)}(b)$,

$$\int_b^c f^{(n+1)}(x)\,dx = (c-b) f^{(n)}(c).$$

In particular, if $f^{(n)}(c) = f^{(n)}(b) = 0$, then must

$$f^{(n+1)}(u) = 0, \qquad b < u < c.$$

It further follows from Art. 9 that, if the first n derivatives of $f(x)$ vanish at $x = b$, then must the first n differells vanish at $x = b$. Conversely, by Art. 11, if the first n differells vanish, or have equal values, at $x = b$, then must the first n derivatives vanish at $x = b$.

Again, if the first n derivatives or differells vanish at $x = b$, we have by Arts. 9, 11,

$$f^{(n+1)}(b) = (b-a)^n f^{n+1}(b).$$

As x converges to the base a, then $f^{(n)}(x)$ converges to the value $f'(a)$.

16. Let $\varphi(x)$ and $\psi(x)$ be two functions which vanish together with their first n derivatives or differells, when $x = b$. Then

$$\mathop{\pounds}_{x=b} \frac{\varphi(x)}{\psi(x)} = \frac{\varphi^{n+1}(b)}{\psi^{n+1}(b)} = \frac{\varphi^{(n+1)}(b)}{\psi^{(n+1)}(b)}.$$

17. Let $F(x)$ and $F_1(x)$ be two functions which vanish, as well as their first n differells, for $x = c$. Then the function

$$J(x) = F_1(x_0)\,F(x) - F_1(x)\,F(x_0)$$

vanishes when $x = x_0$ and $x = c$, and also its first n differells and derivatives vanish for $x = c$.

Then must $J^{(1)}(x) = 0$, for $x = u_1$ between x_0 and c; and $J^{(2)}(x) = 0$, for $x = u_2$ between u_1 and c; and so on, until $J^{(n+1)}(x) = 0$ for $x = u$ between x_0 and c. The same may be said of the first $n + 1$ derivatives of $J(x)$. Hence we have

$$J^{(n+1)}(x) = F_1(x_0)\,F^{(n+1)}(u) - F_1^{(n+1)}(u)\,F(x_0) = 0\,.$$

If $F_1^{(n+1)}(u)$ or $F_1^{n+1}(u')$ is not zero between x_0 and c, we have, dropping the subscript zero,

$$F(x) = F_1(x)\,\frac{F^{(n+1)}(u)}{F_1^{(n+1)}(u)}$$

$$= F_1(x)\,\frac{F^{n+1}(u')}{F^{n+1}(u')}\,,$$

wherein u and u' are values of x which lie somewhere between x and c, and therefore between a and c.

Now let

$$F_1(x) = (x - a)^{n+1}\,.$$

Then

$$F(x) = \frac{x - a}{(n + 1)^{n+1}}\left[\frac{x - a}{u - a}\right]^n F^{(n+1)}(u)$$

$$= \frac{(x - a)^{n+1}}{(n + 1)!}\,F^{n+1}(u')\,,$$

for all values of x throughout the interval (ac). The interval (ac) being such that $a < c$, and u being confined to the interval $(x < c)$, the ratio

$$(x - a)/(u - a) < 1.$$

18. We now proceed to form a series which may be regarded as the fundamental series of this calculus, corresponding to the Taylor series of the Differential Calculus, and we shall derive it from the Generalized Difference Calculus in the same way that the Taylor series is derived from the ordinary Difference Calculus.

Remembering that

$$\delta_h^n = (1 - \varepsilon_h)^n\,, \quad \varepsilon_h^n = (1 - \delta\,)^n\,;$$

we have

$$\begin{aligned} f(x_{g+nh}) &= \varepsilon_h^n f(x_g) \\ &= (1-\delta_h)^n f(x_g) \\ &= f(x_g) - C_{n,1}\delta^1{}_h f(x_g) + C_{n,2}\delta^2{}_h f(x_g) - \ldots + (-1)^n \delta_h^n f(x_g). \end{aligned}$$

Let the law of distribution be $x_{g+rh} = x_g(1-h)^r$, and $nh = k$, so that when $n = \infty$, $h = 0$. Then $x_{g+nh} = x_{g+k}$.

Since, $\delta_h^r x_g = x_g h^r$, we have

$$\begin{aligned} C_{n,r}\delta_h^r f(x_g) &= \frac{n(n-1)\ldots(n-r+1)}{r!} x_g h^r \frac{\delta_h^r f(x_g)}{\delta_h^r x_g} \\ &= x_g (nh)^r \frac{\left[1-\frac{1}{n}\right]\ldots\left[1-\frac{r+1}{n}\right]}{r!} \frac{\delta_h^r f(x_g)}{\delta_h^r x_g} \\ &= x_g \frac{k^r}{r!} f^r(x_g), \end{aligned}$$

when $n = \infty$. Also, we have,

$$\begin{aligned} x_{g+nh} = x_{g+k} &= x_g \left[1-\frac{k}{n}\right]^n, \\ &= x_g e^{-k}, \end{aligned}$$

when $n = \infty$. Therefore, if convergent, we have the series

$$f(ce^{-k}) = f(c) + c\sum_1^\infty (-1)^r \frac{k^r}{r!} f^{(r)}(c),$$

wherein we have written c for x_g. Changing the sign of k, this becomes

$$f(ce^k) = f(c) + c\sum_1^\infty \frac{k^r}{r!} f^{(r)}(c).$$

Let $x = ce^k$, then $k = \log x - \log c$, and

$$f(x) = f(c) + c\sum_1^\infty \frac{(\log x - \log c)^r}{r!} f^{(r)}(c).$$

When the derivatives of $f(x)$ exist, we can write the coefficients in terms of the derivatives at c, § 9, and have*

$$f(x) = f(c) + c\sum_{r=1}^\infty \frac{(\log x - \log c)^r}{r!} \sum_{p=1}^r \Delta^p 0^r \,.\, \frac{(c-a)^p}{p!} f^p(c).$$

* Herschel's theorem. See also Dr. McClintock's Calculus of Enlargement.

Putting $c = 1$, this series furnishes the means of defining the functions $f^{(r)}(c)$, as the coefficients of $(\log x)^r/r!$ in the expansion of $f(x)$ according to the ascending powers of the logarithm of the variable, as the derivatives have been defined as the coefficients in Taylor's series.

The remainder after n terms of this series is easily found, as follows:—

We have

$$\left[\frac{d}{dk}\right] f(ce^k) = x_{g+k} f^{(n)}(x_{g+k}),$$

and when $k = 0$, this becomes

$$\left[\frac{d}{dk}\right]_{k=0} f(c) = cf^{(n)}(c).$$

Let

$$F(k) = f(ce^k) - f(c) - c\sum_1^n \frac{k^r}{r!} f^{(r)}(c).$$

This function vanishes when $k = 0$. Moreover if its first n derivatives are finite, uniform and continuous, they all vanish when $k = 0$, in virtue of the relation exhibited above.

The function

$$J(k) = k^{n+1} F(k_0) - k_0^{n+1} F(k)$$

vanishes when $k = k_0$, and the function and its n derivatives vanish when $k = 0$. Therefore, by § 17,

$$F(k_0) = \frac{k_0^{n+1}}{(n+1)!} F^{n+1}(u). \qquad 0 < u < k$$

Dropping the subscript, we have for the remainder of the series,

$$R = \frac{k^{n+1}}{(n+1)!} \left[\frac{d}{dk}\right]_{k=u}^{n+1} f(ce^u),$$

$$= \frac{k^{n+1}}{(n+1)!} uf^{(n+1)}(u),$$

$$= \frac{k^{n+1}}{(n+1)!} u \sum_1^{n+1} \varDelta^r 0^{n+1} \cdot \frac{(u-a)^r}{r!} f^r(u).$$

19. The fundamental general theorem for the expansion of functions in the Differell Calculus, is this: The function

$$F(x) = \frac{\left| f_0(x), f_1(x_g), f_2^{(1)}(x_g), \ldots, f_{n+1}^{(n)}(x_g) \right|}{\left| f_1(x_g), f_2^{(1)}(x_g), \ldots, f_{n+1}^{(n)}(x_g) \right|},$$

$$= f_0(x) - \sum_1^{n+1} A_r f_r(x),$$

wherein the A's, coefficients of $f_r(x)$, are independent of x but contain the differells of $f(x)$ at x_g, vanishes as do its first n differells when $x = x_g$. Let $F_1(x) = (x - x_g)^{n+1}$. Then by § 17,

$$F(x) = \frac{(x - x_g)^{n+1}}{(n+1)!} F^{n+1}(u).$$

Or, putting c for x_g,

$$f(x) = \sum_1^{n+1} A_r f_r(x) + \frac{(x-c)^{n+1}}{(n+1)!} \left[f^{n+1}(u) - \sum_1^{n+1} A_r f_r^{n+1}(u) \right].$$

20. We give an independent proof of this theorem as follows. For brevity, let

$$\varepsilon_h^n f_m(x_g) = \varepsilon_m^n ; \quad \delta_h^n f_m(x_g) = \delta_m^n .$$

Consider the function

$$F(x) = \frac{\begin{vmatrix} f_0(x), & \ldots, & f_{n+1}(x) \\ \varepsilon_0^0, & \ldots, & \varepsilon_{n+1}^0 \\ \cdot & \cdot & \cdot \\ \varepsilon_0^n, & \ldots, & \varepsilon_{n+1}^n \end{vmatrix}}{\begin{vmatrix} \varepsilon_1^0, & \varepsilon_2^1, & \ldots, & \varepsilon_{n+1}^n \end{vmatrix}}$$

which vanishes when x takes each of the $n + 1$ values x_{g+rh} $(r = 0, 1, 2, \ldots, n)$. The value of this function is, by a theorem of the Differential Calculus (Annals Math., viii, 74),

$$F(x) = (x - x_g)(x - x_{g+h}) \ldots (x - x_{g+nh}) \frac{F^{n+1}(u)}{(n+1)!},$$

wherein u lies between the greatest and least of the values x and x_{g+rh} $(r = 1, \ldots, n)$.

Since we do not change the value of a determinant by subtracting rows from rows, we have

$$F(x) = \frac{\begin{vmatrix} f_0(x), & f_1(x), & \ldots, & f_{n+1}(x) \\ \delta_0^0, & \delta_1^0, & \ldots, & \delta_{n+1}^0 \\ \cdot & \cdot & \cdot & \cdot \\ \delta_0^n, & \delta_1^n, & \ldots, & \delta_{n+1}^n \end{vmatrix}}{\begin{vmatrix} \delta_1^0, & \delta_2^1, & \ldots, & \delta_{n+1}^n \end{vmatrix}}.$$

We still do not change the value of this function when we divide the row of

rth generalized differences in the numerator and denominator of the ratio by $\delta_h^r x_g$ for $r = 1, \ldots, n$, which expresses the terms in these rows as generalized difference-ratios. Letting h converge to zero we pass at once to the result of the previous article.

21. *Forms of Rational Integer Power Series.* Let $f_0(x) = f(x)$ and $f_r(x) = (x - a)^{r-1}$, then we have.

$$\frac{\begin{vmatrix} f(x) & 1, & (x-a), & \dfrac{(x-a)^2}{(c-a)}, & \ldots, & \dfrac{(x-a)^n}{(c-a)^{n-1}} \\ f(c) & 1, & c-a & , c-a & , \ldots, & c-a \\ f^{(1)}(c) & 0, & 1 & , 2^1 & , \ldots, & n^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(n)}(c) & 0, & 1 & , 2^n & , \ldots, & n^n \end{vmatrix}}{|\,1,\ 2^2,\ \ldots,\ n^n\,|} = \frac{(x-c)^{n+1}}{(n+1)!} f^{n+1}(u),$$

wherein u lies between x and c. We have therefore

$$f(x) = A_0 + \sum_{p=1}^{n} (-1)^{p+1} A_p \frac{(x-a)^p}{(c-a)^{p-1}} + \frac{(x-c)^{n+1}}{(n+1)!} f^{n+1}(u),$$

wherein

$$A_p = \sum_{r=1}^{n} (-1)^{r+1} \frac{\Delta(r,p)}{\Delta} f^{(r)}(c).$$

$\Delta(r, p)$ is the determinant $\Delta = |\,1^0, 2^2, \ldots, n^n\,|$, with its pth column and rth row deleted. The *remainder* of the series vanishes when $n = \infty$.

With the same notations as before, except now let

$$f_r(x) = \frac{1}{(r-1)!} \frac{(x-c)^{r-1}}{(c-a)^{r-2}},$$

we have

$$(x-a)\left[\frac{d^n}{d^n x}\right](x-c)^p = \sum_{1}^{n} \Delta^r 0^n \,.\, \frac{(x-a)^r}{r!} \left[\frac{d}{dx}\right]^r (x-c)^p$$

$$= \sum_{1}^{p} \Delta^r 0^n \,.\, C_{p,r} (x-a)^r (x-c)^{p-r}.$$

If $x = c$, then

$$\left[\frac{d}{d^n x}\right]^n_{x=c} (x-c)^p = \Delta^p 0^n \,.\, (c-a)^{p-1}.$$

Moreover $\Delta^p 0^n = 0$ for $p > n$ and is $p!$ when $n = p$. Therefore

$$\begin{vmatrix} f(x) \,, & 1\,, & \frac{1}{1!}\frac{(x-c)^1}{(c-a)^0}\,, & \frac{1}{2!}\frac{(x-c)^2}{(c-a)^1}\,, & \ldots\,, & \frac{1}{n!}\frac{(x-c)^n}{(c-a)^{n-1}} \\ f(c)\,, & 1\,, & 0\,, & 0\,, & \ldots\,, & 0 \\ f^{(1)}(c)\,, & 0\,, & 1\,, & 0\,, & \ldots\,, & 0 \\ f^{(2)}(c)\,, & 0\,, & \frac{\Delta^1 0^2}{1!}\,, & 1\,, & \ldots\,, & 0 \\ f^{(3)}(c)\,, & 0\,, & \frac{\Delta^1 0^3}{1!}\,, & \frac{\Delta^2 0^3}{2!}\,, & \ldots\,, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f^{(n)}(c)\,, & 0\,, & \frac{\Delta^1 0^n}{1!}\,, & \frac{\Delta^2 0^n}{2!}\,, & \ldots\,, & 1 \end{vmatrix} = \frac{(x-c)^{n+1}}{(n+1)!} f^{(n+1)}(u).$$

The coefficients of $(x - c)^r$ are now in finite form. We are now prepared to give the general law promised in § 11, for the coefficient of $(x - c)^r$ in this series must equal the coefficient of the like power in Taylor's, therefore we have

$$(-1)^r (c-a)^{r-1} f^r(c) = \begin{vmatrix} f^{(1)}(c)\,, & 1\,, & 0\,, & \ldots\,, & 0 \\ f^{(2)}(c)\,, & \frac{\Delta^1 0^2}{1!}\,, & 1\,, & \ldots\,, & 0 \\ f^{(3)}(c)\,, & \frac{\Delta^1 0^3}{1!}\,, & \frac{\Delta^2 0^3}{2!}\,, & \ldots\,, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f^{(r)}(c)\,, & \frac{\Delta^1 0^r}{1!}\,, & \frac{\Delta^2 0^r}{2!}\,, & \ldots\,, & \frac{\Delta^{r-1} 0^r}{(r-1)!} \end{vmatrix}$$

If $c = a$, $f^{(r)}(c) = f^r(a)$, and the determinant vanishes as was stated in § 11.

22. When $f(x)$ is a Taylor's function, we have, base zero,

$$\left[\frac{d^n}{d^n x}\right] f(x) = \sum_{r=1}^{\infty} r^n x^{r-1} \frac{f^r(0)}{r!} = \sum_{1}^{n} \Delta^r 0^n \,.\, x^{r-1} \frac{f^r(x)}{r!}\,.$$

Put $f(x) = e^x$, then

$$e^x \sum_{1}^{n} \Delta^r 0^n \,.\, \frac{x^{r-1}}{r!} = 1 + \frac{2^n}{2!} x + \frac{3^n}{3!} x^2 + \ldots$$

or the series on the right is the product of e^x and a rational integral. If $x = 1$,

$$e \sum_1^n \frac{\Delta^r 0^n}{r!} = 1 + \frac{2^n}{2!} + \frac{3^n}{3!} + \dots .$$

By assigning values to n we get different infinite series for e.

23. We have heretofore been using $f(x)$ as a Taylor's series. If the function cannot be expressed in Taylor's series, we must go back to the fundamental limiting ratio to determine the differells. For example, when the base is zero, $(\log x)^m$ cannot be expressed by Maclaurin's series. Its nth differell, however, is readily found to be

$$\left[\frac{d^n}{d^n x}\right] (\log x)^m = m(m-1)\dots(m-n+1)\,\frac{(\log x)^{m-n}}{x}.$$

For

$$\left[\frac{d^n}{d^n x}\right] (\log x)^m = \frac{1}{x}\left[x\frac{d}{dx}\right]^n (\log x)^m$$

$$= \frac{1}{x}\left[\frac{d}{d(\log x)}\right]^n \cdot (\log x)^m .$$

In like manner when the base is a, we have

$$\left[\frac{d^n}{d^n x}\right] [\log(x-a)]^m = \frac{1}{x-a}\left[\frac{d}{d\log(x-a)}\right]^n [\log(x-a)]^m$$

$$= \frac{m!}{n!}\,\frac{[\log(x-a)]^{m-n}}{x-a}.$$

We can now easily deduce the series of § 18, by the general method of §§ 19, 20.

Again, letting $f(x) = \log x$, we have

$$\sum_1^n (-1)^{r-1} \frac{\Delta^r 0^n}{r}$$

equal to unity for $n = 1$ and zero for $n = 1, 2, 3, \dots$.

IV. (*b*) Integrells or Negative Differells.*

24. Heretofore we have considered only positive values of n in using differells, we propose now to notice those corresponding functions in which n is negative. We shall, for brevity, call such functions *integrells*. The integrell bearing to the differell a relation somewhat analogous to that of the integral to the derivative.

* Read before The American Mathematical Society at Springfield, Mass., Aug., 1895.

25. We might simply continue to define negative differells by using the property that each differell is the mean value of the one of next higher order. Whence, descending by units,

$$f^{(0)}(x) = \frac{1}{x-a}\int_a^x f^{(1)}(x)\,dx\,,$$

$$= \frac{f(x) - f(a)}{x-a}\,.$$

Which is the fundamental function with which this calculus deals. Geometrically pictured it represents the inclination of the secant through the points $[a, f(a)]$, $[x, f(x)]$ to the x-axis. This becomes the tangent to the curve $y = f(x)$ at $x = a$. Correspondingly

$$f^{(-1)}(x) = \frac{1}{x-a}\int_a^x f^{(0)}(x)\,dx\,,$$

is the mean inclination of the secant for all points in the interval (ax). Generally, we have

$$f^{(-n-1)}(x) = \frac{1}{x-a}\int_a^x f^{(-n)}(x)\,dx\,.$$

Or, the integrell of any order is the mean value of the integrell of next lower order. Clearly, if c is a constant,

$$\left[\frac{d^{-1}}{d^{-1}x}\right] c = 0\,.$$

If $f(x) = (x-a)^m$, it follows directly that

$$\left[\frac{d^{-n}}{d^{-n}x}\right](x-a)^m = \frac{(x-a)^{m-1}}{m^n}\,.$$

If zero be the base we have the corresponding values as for differells. Therefore, so far as Taylor functions are concerned the differells derived are true for positive and negative integral indices.

26. But, it is well to found the integrell as we did the differell, upon the limiting ratio. We have according to our original convention

$$\delta_h^{-n}\,.\,f(x_g) = (1 - \varepsilon_h)^{-n}\,.\,f(x_g)$$

$$= \sum_{r=0}^{\infty} H_n^r\,.\,f(x_{g+rh})\,,$$

wherein $H_n^0 = 1$, and

$$H_n^r = \frac{1}{r!}\, n(n+1)\ldots(n+r-1).$$

Since $(1-\varepsilon_h)^{-n}$ is expressed as an infinite series we must give this definite meaning before proceeding further. What we are to be concerned with, is the limiting ratio

$$\mathop{\pounds}_{h=0} \frac{\delta_h^{-n} f(x_g)}{\delta_h^{-n} x_g} = \mathop{\pounds}_{h=0} \frac{(1-\varepsilon_h)^{-n} f(x_g)}{(1-\varepsilon_h)^{-n} x_g}.$$

We have, when the distribution law is $x_{g-rh} = x_g e^{-rh}$,

$$\delta_h^{-n} x_g = x_g \left[1 + ne^{-h} + \frac{1}{2!} n(n+1) e^{-2h} + \ldots\right]$$

$$= x_g (1 - e^{-h})^{-n},$$

$$\delta_h^{-n} x_g^m = x_g^m (1 - e^{-mh})^{-n}.$$

And these are true whether n is integral or fractional. Consequently, we have

$$\left[\frac{d^{-n}}{d^{-n}x_g}\right] x_g^m = \mathop{\pounds}_{h=0} \frac{\delta_h^{-n} x_g^m}{\delta_h^{-n} x_g} = x_g^{m-1} \mathop{\pounds}_{h=0} \left[\frac{1-e^{-h}}{1-e^{-mh}}\right]^n$$

$$= \frac{x^{m-1}}{m^n}.$$

But when we attempt to form the integrells of an absolute constant by this method, we are met by the difficulty that the result is infinity. For $\delta^n c = 0$, and

$$\varepsilon^n c = (1-\delta)^n c = c.$$

$$\therefore \qquad \delta^{-n} c = (1-\varepsilon)^{-n} c = c(1-1)^{-n} = \infty,$$

while

$$\delta^{-n} x = (1 - e^{-h})^{-n}.$$

Therefore we infer that the integrell operation applies only to functions which have no absolute term, that is to say to the difference of two values of a function.

We have

$$f(x+a) - f(a) = xf'(a) + \frac{x^2}{2!} f''(a) + \cdots,$$

and

$$\left[\frac{d^{-n}}{d^{-n}x}\right] [f(x+a) - f(a)] = f'(a) + \frac{x}{2!\,2^n} f''(a) + \frac{x^2}{3!\,3^n} f'''(a) + \cdots.$$

Now if we put $n = 0$, we define the integrell of zero order to be

$$\left[\frac{d^{-0}}{d^{-0}x}\right][f(x+a)-f(a)] = f'(a) + \frac{x}{2!}f''(a) + \frac{x^2}{3!}f'''(a) + \ldots,$$

$$= \frac{f(x+a)-f(a)}{x}.$$

Or, changing $x + a$ into x,

$$\left[\frac{d^{-n}}{d^{-n}x}\right][f(x)-f(a)] = f'(a) + \frac{x-a}{2!\,2^n}f''(a) + \ldots$$

$$\left[\frac{d^{-0}}{d^{-0}x}\right][f(x)-f(a)] = \frac{f(x)-f(a)}{x-a},$$

which we agree to write $f^{(0)}(x)$, also the nth integrell of the above function we agree to write $f^{(-n)}(x)$. Integrating $f^{(-n)}(x)$ between a and x, we find

$$f^{(-n-1)}(x) = \frac{1}{x-a}\int_a^x f^{(-n)}(x)\,dx.$$

Therefore the integrell is the mean value, between x and a, of the integrell of next lower order, and each has distinct reference to an interval whose base is a. The integrell of a constant is thus zero. The foundation of the integrells as here developed is the same as that of the preceding article.

27. Applying the above results to the function $(x-a)^m$, base a, we have for the nth integrell

$$\frac{(x-a)^{m-1}}{m^n}.$$

In like manner when c is not the base a, we have when $f(x) = (x-c)^m$,

$$f(x) - f(a) = C_{m,1}(a-c)^{m-1}(x-a) + C_{m,2}(a-c)^{m-2}(x-a)^2 + \ldots + C_{m,m}(x-a)^m.$$

Therefore

$$f^{(-n)}(x) = C_{m,1}(a-c)^{m-1} + C_{m,2}(a-c)^{m-2}\frac{(x-a)}{2^n} + \ldots + C_{m,m}\frac{(x-a)^{m-1}}{m^n}.$$

In this put $x = c$, then

$$f^{(-n)}(c) = m(a-c)^{m-1}\left[1 - \frac{C_{m-1,1}}{2^{n+1}} + \ldots + (-1)^{m-1}\frac{C_{m-1,m-1}}{m^{n+1}}\right].$$

Now we have (Chrystal's Algebra ii, 183),

$$\frac{n!}{x(x+1)\ldots(x+n)}P_p^n = \sum_{r=0}^{n}(-1)^r\frac{C_{n,r}}{(x+r)^{p+1}}.$$

Let $x = 1$, $n = m - 1$, $p = n$, and we have

$$1 - \frac{C_{m-1,1}}{2^{n+1}} + \ldots + (-1)^{m-1}\frac{C_{m-1,m-1}}{m^{n+1}} = \frac{1}{m} P_n^m,$$

wherein P_n^m is the sum of the products of the numbers

$$\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{m}$$

taken n at a time with repetition. Therefore

$$f^{(-n)}(c) = (a - c)^{m-1} P_n^m .$$

28. Expansion of Functions:

We have

$$\varepsilon_h^{-n} = (1 - \delta_h)^{-n},$$

therefore

$$\varepsilon_h^{-n} f(x_g) = f(x_g) + H_n^1 \delta_h^1 f(x_g) + H_n^2 \delta_h^2 f(x_g) + \ldots$$

or,

$$f(x_{g-nh}) = f(x_g) + x_g \left\{ (nh)\frac{\delta^1 f}{\delta^1 x} + \frac{(nh)^2}{2!}\left[1 + \frac{1}{n}\right]\frac{\delta^2 f}{\delta^2 x} + \ldots \right\},$$

wherein $x_{g-rh} = x_g(1 - h)^r$. Let $nh = k$, and h converge to zero. Then

$$f(x_{g-k}) = f(x_g) + x_g \left\{ k f^{(1)}(x_g) + \frac{k^2}{2!} f^{(2)}(x_g) + \ldots \right\}$$

which is the same logarithmic expansion as was deduced in § 18.

29. If $F(x)$ be a uniform, finite, and continuous function from a to c, and if the first n integrells of $F(x)$ vanish at $x = c$, then must the function $F(x)$ have $n + 1$ distinct equal values between a and c, including these limits. We have

$$F^{(-n)}(c) = \frac{1}{c - a}\int_a^c F^{(-n+1)}(x)\, dx = 0,$$

Consequently

$$F^{(-n+1)}(u_1) = 0. \qquad a < u_1 < c$$

But since

$$\int_a^c = \int_a^{u_1} + \int_{u_1}^c,$$

then must

$$F^{(-n+2)}(u_2) = 0; \quad F^{(-n+2)}(u_3) = 0,$$

$(a < u_2 < u_1)$, $(u_1 < u_3 < c)$. In like manner must $F^{(-n+3)}(x) = 0$, between

u_2 and a, u_2 and u_3, u_3 and c, and also at c. Continuing thus we see that $F^{(-n+r)}(x)$ must vanish for $(r+1)$ distinct values of x between a and c, including c. Therefore $F^{(-1)}(x)=0$, for n distinct values of x between a and c, including c. But

$$F^{(-1)}(x)=\frac{1}{x-a}\int_a^x \frac{F(x)-F(a)}{x-a}\,dx\,.$$

It follows that $F(x)=F(a)$ for these n values of x. Consequently $F(x)$ has $n+1$ distinct equal values between a and c, inclusive.

30. If $F(x)$ be a Taylor's function in the interval (a, c), it has been shown in the Differential Calculus (Annals Math., viii, 74), that when it has $n+1$ equal values in the interval, we have.

$$F(x)-F(c)=(x-c)\,(x-u_1)\ldots(x-u_{n-1})\,(x-a)\,\frac{F^{n+1}(u)}{(n+1)!}\,,$$

wherein u lies between the greatest and least of a, c, $u_1, \ldots, u_{n-1}$, or in the interval (ac). We may write this result

$$F(x)-F(c)=\theta^{n+1}\,\frac{(a-c)^{n+1}}{(n+1)!}\,F^{n+1}(u)$$

wherein $0<\theta<1$.

31. THEOREM.

$$F(x)=\frac{\begin{vmatrix} f(x) & , f_1(x) & , \ldots, & f_{n+1}(x) \\ f(c) & , f_1(c) & , \ldots, & f_{n+1}(c) \\ f^{(-1)}(c), & f_1^{(-1)}(c), & \ldots, & f_{n+1}^{(-1)}(c) \\ \cdot & \cdot & \cdot & \cdot \\ f^{(-n)}(c), & f_1^{(-n)}(c), & \ldots, & f_{n+1}^{(-n)}(c) \end{vmatrix}}{[\text{minor of } f(x)]}$$

is such that its first n integrells and the function vanish at $x=c$. Therefore the function vanishes $n+1$ times between a and c, inclusive, and by § 29, we have

$$F(x)=\theta^{n+1}\,\frac{(c-a)^{n+1}}{(n+1)!}\,F^{n+1}(u)\,.$$

Moreover, since the function also vanishes at $x=a$, we can replace c by a in the second row of the numerator.

32. Let $f_{r+1}(x) = (x-a)^r$, then $F^{n+1}(u) = f^{n+1}(u)$, and

$$\frac{\begin{vmatrix} \dfrac{f(x)-f(a)}{x-a}, & 1, & \dfrac{x-a}{c-a}, & \ldots, & \left[\dfrac{x-a}{c-a}\right]^{n-1} \\ f^{(-1)}(c) & , 1, & 2^{-1} & , \ldots, & n^{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(-n)}(c) & , 1, & 2^{-n} & , \ldots, & n^{-n} \end{vmatrix}}{\left| 1^0, 2^{-2}, \ldots, n^{-n} \right|} = \theta^{n+1}\frac{(c-a)^{n+1}}{(n+1!)} f^{n+1}(u).$$

Again, let $f_{r+1}(x) = (x-c)^r$, then as before $F^{n+1}(u) = f^{n+1}(u)$ and

$$\frac{\begin{vmatrix} \dfrac{f(x)-f(c)}{x-c}, & 1, & \dfrac{x-c}{a-c}, & \ldots, & \left[\dfrac{x-c}{a-c}\right]^{n-1} \\ f^{(-1)}(c) & , 1, & P_1^1 & , \ldots, & P_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(-n)}(c) & , 1, & P_1^n & , \ldots, & P_n^n \end{vmatrix}}{\left| 1, P_2^2, \ldots, P_n^n \right|} = \theta^{n+1}\frac{(c-a)^{n+1}}{(n+1)!} f^{n+1}(u).$$

These series cannot be written in practicable form until the ratio $\Delta(r, p)/\Delta$ can be evaluated when $n = \infty$, as was the case in § 21.

V. Generalized Difference Ratios.

33. In forming the so-called generalized differences of a function, we departed from the ordinary method by changing the distribution of the argument, but retained the method of operation of differencing by forming the successive differences in the ordinary manner. We might have, more generally still, defined the nth generalized difference of a function at x_g to have been

$$\delta_h^n f(x_g) = \begin{vmatrix} f(x_g) & , 1, & p_0^1, & \ldots, & p_0^{n-1} \\ f(x_{g+h}) & , 1, & p_1^1, & \ldots, & p_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(x_{g+nh}), & 1, & p_n^1, & \ldots, & p_n^{n-1} \end{vmatrix} \div \left| p_1^0, p_2^1, \ldots, p_n^{n-1} \right|,$$

wherein the p's are numbers ($p_0 = 0$), the law of whose formation is arbitrary. The corresponding difference of the argument is obtained by deleting the f. These differences evidently vanish when $h = 0$.

We have by expansion

$$\delta_h^n f(x_g) = \sum_0^n (-1)^r A_r f(x_{g+rh}),$$

wherein $A_0 = 1$, and

$$A_r = \frac{(p_n - p_0) \ldots (p_1 - p_0)}{(p_n - p_r) \ldots (p_{r+1} - p_r)(p_r - p_{r-1}) \ldots (p_r - p_0)},$$

with a corresponding value for $\delta_h^n x_g$. We are concerned with the limiting ratio

$$\mathop{\pounds}_{h=0} \frac{\delta_h^n f(x_g)}{\delta_h^n x_g}.$$

If the law of the p's is $p_r = r$, we have the results of § 2.

34. Let the law of the p's be not $p_r = r$, and let the law of distribution be

$$x_{g+rh} = x_g + rh.$$

Then evidently

$$\begin{aligned} \mathop{\pounds}_{h=0} \frac{\delta_h^n f(x_g)}{\delta_h^n x_g} &= \mathop{\pounds}_{h=0} \frac{\sum_0^n (-1)^r A_r f(x_g + rh)}{h \sum_1^n (-1)^r A_r}, \\ &= \mathop{\pounds}_{h=0} \frac{\sum_1^n (-1)^r rA_r f'(x_{g+rh})}{\sum_1^n (-1)^r rA_r}, \\ &= f'(x_g). \end{aligned}$$

And generally

$$f^{m+1}(x_g) = \mathop{\pounds}_{h=0} \frac{\delta_h^n f^m(x_g)}{\delta_h^n x_g},$$

whatever be the value of the finite positive integer n.

If the law of the p's is not $p_r = \varphi(r)$, the same results hold when

$$x_{g+rh} = x_g + h\,\varphi(r),$$

as for example $\varphi(r) = 1/r$.

35. If the law is $x_{g+rh} = x_g e^{p_r h}$, we have the Calculus of Multiplication, as is easily verified by

$$\mathop{\pounds}_{h=0} \frac{\delta_h^n x^m}{\delta_h^n x} = m^n x^{m-1}.$$

ON THE SYMMETRICAL FORM OF THE DIFFERENTIAL EQUATIONS OF PLANETARY MOTIONS.

By Prof. Ormond Stone, Charlottesville, Va.

The differential equations usually employed in determining the motions of the planets,

$$\begin{aligned}\frac{d^2x_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3}\,x_i &= \frac{\partial R_i}{\partial x_i},\\ \frac{d^2y_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3}\,y_i &= \frac{\partial R_i}{\partial y_i}, \qquad i = 1, 2, \ldots, n. \quad (1)\\ \frac{d^2z_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3}\,z_i &= \frac{\partial R_i}{\partial z_i},\end{aligned}$$

are unsymmetrical, since the functions $R_1, R_2, \ldots, R_n$ are not the same for each planet. The symmetrical form* may be obtained in the following manner:—

1. Let ξ_i, η_i, ζ_i be the coordinates of the different masses of the system M_i referred to fixed rectangular axes. Let G_i be the center of gravity of the masses $M_1, M_2, \ldots, M_i$; let x_i, y_i, z_i be the coordinates of M_i referred to three axes parallel to the fixed axes, but passing through G_{i-1}; let X_i, Y_i, Z_i, be the coordinates of G_i; $\mu_i = \sum_1^i {}_\sigma m_\sigma$. If the number of the bodies be n, G_n will be the center of gravity of the system. We shall also have

$$\xi_i = X_{i-1} + x_i, \qquad (2)$$

in which $X_1 = \xi_1$; and, if we put $x_1 = 0$, we have $X_0 = \xi_1$, in which X_0 is introduced merely in order that the nomenclature in (2) may be applicable throughout.

We have, also, in accordance with the properties of centers of gravity,

$$\mu_{i-1}X_{i-1} + m_i\xi_i = \mu_i X_i; \qquad (3)$$

whence, substituting for ξ_i its value as given by (2), we obtain

$$\mu_i X_{i-1} + m_i x_i = \mu_i X_i,$$

or

$$m_i x_i = \mu_i (X_i - X_{i-1}). \qquad (4)$$

* See Tisserand's Mécanique Céleste, t. i, chap. iv, the substance of which is derived from an interesting memoir by M. R. Radau, entitled "Sur une transformation des équations differentielles de la Dynamique" (Annales de l'École Normale, 1re série, t. v).

Squaring (2) and multiplying by m_i, we have

$$m_i\xi_i^2 = m_i x_i^2 + 2m_i x_i X_{i-1} + m_i X_{i-1}^2;$$

whence, adding $m_i x_i (X_i - X_{i-1}) - \dfrac{m_i^2}{\mu_i} x_i^2 = 0$,

$$m_i\xi_i^2 = m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + m_i x_i (X_i + X_{i-1}) + m_i X_{i-1}^2.$$

If $\mu_i (X_i - X_{i-1})$ be substituted for $m_i x_i$ in the middle term, this becomes

$$m_i\xi_i^2 = m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_i (X_i^2 - X_{i-1}^2) + m_i X_{i-1}^2$$

$$= m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2;$$

or, since $\mu_0 = 0$,

$$\overset{n}{\underset{1}{\Sigma}}{}_i\, m_i\xi_i^2 = \overset{n}{\underset{1}{\Sigma}}{}_i\, m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_n X_n^2. \tag{5}$$

2. Equation (5) may be obtained in a still simpler manner. Equation (4) gives

$$X_i = X_{i-1} + \frac{m_i}{\mu_i} x_i;$$

whence, substituting in (2), we have

$$\xi_i = X_i + \frac{\mu_{i-1}}{\mu_i} x_i. \tag{6}$$

From (3) we have, also,

$$m_i\xi_i = \mu_i X_i - \mu_{i-1} X_{i-1}. \tag{7}$$

Multiplying the left hand member by ξ_i, and the terms of the right hand member by the values of ξ_i given by (6) and (2), respectively, equation (7) becomes

$$m_i\xi_i^2 = \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 + \mu_{i-1} x_i (X_i - X_{i-1})$$

$$= \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 + m_i \frac{\mu_{i-1}}{\mu_i} x_i^2;$$

whence, as before,

$$\overset{n}{\underset{1}{\Sigma}}{}_i\, m_i\xi_i^2 = \overset{n}{\underset{1}{\Sigma}}{}_i\, m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_n X_n^2. \tag{5}$$

3. If we differentiate (2) and (3) with reference to t, we see at once that the relations between the differentials are exactly the same as the relations

between the corresponding variables; hence we may substitute the differentials for the variables in (5), and obtain

$$\sum_1^n{}_i\, m_i \left[\frac{d\xi_i}{dt}\right]^2 = \sum_1^n{}_i\, m_i \frac{\mu_{i-1}}{\mu_i}\left[\frac{dx_i}{dt}\right]^2 + \mu_n \left[\frac{dX_n}{dt}\right]^2. \tag{8}$$

There are also relations similar to (5) and (8) for the coordinates η and ζ. Adding, we have

$$\sum_1^n{}_i\, m_i\rho_i^2 = \sum_1^n{}_i\, m_i \frac{\mu_{i-1}}{\mu_i} r_i^2 + \mu_n R_n^2, \tag{9}$$

in which $\rho_i^2 = \xi_i^2 + \eta_i^2 + \zeta_i^2$, $r_i^2 = x_i^2 + y_i^2 + z_i^2$, and $R_n^2 = X_n^2 + Y_n^2 + Z_n^2$; also

$$\begin{aligned} 2T &= \sum_1^n{}_i\, m_i \left[\left[\frac{d\xi_i}{dt}\right]^2 + \left[\frac{d\eta_i}{dt}\right]^2 + \left[\frac{d\zeta_i}{dt}\right]^2\right] \\ &= \sum_1^n{}_i\, m_i \frac{\mu_{i-1}}{\mu_i}\left[\left[\frac{dx_i}{dt}\right]^2 + \left[\frac{dy_i}{dt}\right]^2 + \left[\frac{dz_i}{dt}\right]^2\right] \\ &\quad + \mu_n \left[\left[\frac{dX_n}{dt}\right]^2 + \left[\frac{dY_n}{dt}\right]^2 + \left[\frac{dZ_n}{dt}\right]^2\right]. \end{aligned} \tag{10}$$

4. An examination of (6) and (2) shows that we may write

$$\frac{d\eta_i}{dt} = \frac{dY_i}{dt} + \frac{\mu_{i-1}}{\mu_i}\frac{dy_i}{dt} = \frac{dY_{i-1}}{dt} + \frac{dy_i}{dt}. \tag{11}$$

Multiplying $m_i\xi_i$ by the first member of (11), $\mu_i X_i$ by the second, and $\mu_{i-1}X_{i-1}$ by the third, equation (7) becomes

$$\begin{aligned} m_i\xi_i \frac{d\eta_i}{dt} &= \mu_i X_i \frac{dY_i}{dt} + \mu_{i-1}X_i\frac{dy_i}{dt} - \mu_{i-1}X_{i-1}\frac{dY_{i-1}}{dt} - \mu_{i-1}X_{i-1}\frac{dy_i}{dt} \\ &= \mu_i X_i\frac{dY_i}{dt} - \mu_{i-1}X_{i-1}\frac{dY_{i-1}}{dt} + m_i\frac{\mu_{i-1}}{\mu_i}x_i\frac{dy_i}{dt}. \end{aligned}$$

A similar expression can be readily obtained for $m_i\eta_i\dfrac{d\xi_i}{dt}$; whence

$$\begin{aligned} \sum_1^n{}_i\, m_i\left[\xi_i\frac{d\eta_i}{dt} - \eta_i\frac{d\xi_i}{dt}\right] &= \mu_n\left[X_n\frac{dY_n}{dt} - Y_n\frac{dX_n}{dt}\right] \\ &\quad + \sum_1^n{}_i\, m_i\frac{\mu_{i-1}}{\mu_i}\left[x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right]. \end{aligned} \tag{12}$$

5. If U be the force function of the system, we may put

$$P = \frac{\partial T}{\partial \frac{dX_n}{dt}}, \qquad P_1 = \frac{\partial T}{\partial \frac{dY}{dt}}, \qquad P_2 = \frac{\partial T}{\partial \frac{dZ}{\partial t}};$$

$$p_{3i} = \frac{\partial T}{\partial \frac{dx_i}{dt}}, \qquad p_{3i+1} = \frac{\partial T}{\partial \frac{dy_i}{dt}}, \qquad p_{3i+2} = \frac{\partial T}{\partial \frac{dz_i}{dt}};$$

and write the equations of motion in the well-known canonical form,

$$\frac{dX_n}{dt} = \frac{\partial(T-U)}{\partial P}, \ldots; \qquad \frac{dx_i}{dt} = \frac{\partial(T-U)}{\partial p_{3i}}, \ldots;$$

$$\frac{dP}{dt} = -\frac{\partial(T-U)}{\partial X_n}, \ldots; \qquad \frac{dp_{3i}}{dt} = -\frac{\partial(T-U)}{\partial x_i}, \ldots. \tag{13}$$

Since U does not contain X_n, Y_n, Z_n, but only the differences $\xi_i - \xi_j$, $\eta_i - \eta_j$, $\zeta_i - \zeta_j$, equations (10) and (13) give

$$\frac{d^2X_n}{dt^2} = \frac{d^2Y_n}{dt^2} = \frac{d^2Z_n}{dt^2} = 0;$$

whence, by integration,

$$X_n = \alpha t + \alpha', \quad Y_n = \beta t + \beta', \quad Z_n = \gamma t + \gamma', \tag{14}$$

in which α, α', β, β', γ, γ' are arbitrary constants.

Also, since T does not contain the x_i, y_i, z_i, but only the velocities, (10) and (13) give

$$m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i}, \tag{15}$$

which have the symmetrical form desired.

CALCULUS OF VARIATIONS.

By Dr. Harris Hancock, Chicago, Ill.

Second Article.

1. *Discussion of the first variation.* As a more general form of the integrals which were given in problems I, II, III, and IV (see Annals of Mathematics, Vol. IX, No. 6, p. 183 et seq.) let us consider the integral

$$I = \int_{x_0}^{x_1} F(x, y; y')\, dx,$$

where $F(x, y; y')$ is a known function of x, y, and y', and where the limits of this integral, x_1 and x_0, are fixed. Hence (loc. cit., p. 189)

$$\begin{aligned} \Delta I &= \int_{x_0}^{x_1} F(x, y + \varepsilon\eta;\ y' + \varepsilon\eta')\, dx - \int_{x_0}^{x_1} F(x, y;\ y')\, dx \\ &= \int_{x_0}^{x_1} [F(x, y + \varepsilon\eta;\ y' + \varepsilon\eta') - F(x, y;\ y')]\, dx. \end{aligned}$$

This expression, when expanded by Taylor's theorem, is

$$= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\varepsilon\eta + \frac{\partial F}{\partial y'}\varepsilon\eta' + \varepsilon^2(\ ) + \dots \right] dx.$$

We also have, as on p. 189,

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{1\,.\,2}\delta^2 I + \dots;$$

and by comparing the coefficients of ε in these two expressions, it follows that

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right] dx. \tag{A}$$

Remark. In the particular case given (p. 183), $F = 2\pi y\sqrt{1 + y'^2}$. Hence

$$\frac{\partial F}{\partial y} = 2\pi\sqrt{1 + y'^2} \text{ and } \frac{\partial F}{\partial y'} = \frac{2\pi y y'}{\sqrt{1 + y'^2}};$$

and when these relations are substituted in (A), we have, as on p. 190,

$$\delta I = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1 + y'^2}\, \eta + \frac{yy'}{\sqrt{1 + y'^2}} \eta' \right] dx .$$

2. From the relation

$$\Delta I = \varepsilon \delta I + \frac{\varepsilon^2}{1 \,.\, 2} \delta^2 I + \dots ,$$

it is seen that when ε is taken very small, ε^2 is as near as we wish to zero; and consequently when ε is positive and indefinitely small, ΔI is *positive*. On the other hand when ε is indefinitely small and negative, ΔI is *negative*.

Hence the total variation ΔI of the integral will be either positive or negative according as ε is positive or negative, so long as δI is different from zero; and consequently there can be neither a maximum nor a minimum value of the integral.

We know, however, if I is a maximum ΔI is always positive, and if I is a minimum ΔI is always negative; and consequently *in order to have a maximum or a minimum value of the integral δI must be zero.*

3. Applying the above result to the example given in Art. 1, we have

$$0 = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1 + y'^2}\, \eta + \frac{yy'}{\sqrt{1 + y'^2}} \frac{d\eta}{dx} \right] dx . \tag{1}$$

Integrating by parts, the integral

$$\int_{x_0}^{x_1} \frac{yy'}{\sqrt{1 + y'^2}} d\eta = \left[\frac{yy'}{\sqrt{1 + y'^2}} \eta \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \left[\frac{yy'}{\sqrt{1 + y'^2}} \right] \eta dx ;$$

and since, by hypothesis (p. 189), $\eta = 0$ at both of the fixed points P_0 and P_1, we have

$$\left[\frac{yy'}{\sqrt{1 + y'^2}} \eta \right]_{x_0}^{x_1} = 0 .$$

Hence (1) may be written

$$0 = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1 + y'^2} - \frac{d}{dx} \left[\frac{yy'}{\sqrt{1 + y'^2}} \right] \right] \eta dx . \tag{2}$$

4. We assert that in the expression above

$$\sqrt{1 + y'^2} - \frac{d}{dx} \left[\frac{yy'}{\sqrt{1 + y'^2}} \right]$$

must always be zero between the limits x_0 and x_1. For, assuming that the contrary is the case, then, since η is arbitrary, we may, with Heine, write

$$\eta = (x - x_0)(x_1 - x)\left[\sqrt{1+y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right]\right],$$

where η becomes zero for the values $x = x_0$ and $x = x_1$, and is positive within the interval $x_0 \ldots x_1$.

Substituting this value of η in (2) of the preceding section, we have

$$0 = 2\pi\int_{x_0}^{x_1}\left[\sqrt{1+y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right]\right]^2 (x - x_0)(x_1 - x)\,dx, \tag{3}$$

an expression which is positive within the whole interval $x_0 \ldots x_1$.

The integrand in (3), looked upon as a sum of infinitely small elements, has all its elements of the same sign and positive; so that the only possible way for the right hand member of (3) to be zero is that

$$\sqrt{1+y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] = 0.$$

We therefore have a differential equation of the second order for the determination of the unknown quantity y.

5. This differential equation is a special case of the more general differential equation, which may be derived from the integral

$$I = \int_{x_0}^{x_1} F(y, y')\,dx;$$

whence, as before (Arts. 1 and 3),

$$\delta I = \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right]dx$$

$$= \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right]\eta\,dx.$$

And, as above (Art. 3),

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0,$$

i. e.

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]. \tag{1}$$

But

$$dF(y, y') = \frac{\partial F}{\partial y}\, dy + \frac{\partial F}{\partial y'}\, dy', \tag{2}$$

or

$$dF(y, y') - \left[\frac{\partial F}{\partial y}\, dy + \frac{\partial F}{\partial y'}\, dy'\right] = 0 .$$

Hence, from (1),

$$dF(y, y') - \left[\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] dy + \frac{\partial F}{\partial y'}\, dy'\right] = 0$$

or

$$dF(y, y') - d\left\{y' \frac{\partial F}{\partial y'}\right\} = 0 ,$$

and integrating,

$$F(y, y') - y' \frac{\partial F}{\partial y'} = C , \tag{3}$$

where C is the constant of integration.

The relation (3) exists only when the integrand of the given integral does not contain *explicitly* the variable x; otherwise the relation (2) would not be true, and then we could not deduce (3).

6. Applying this relation (3) to the special case above (Art. 4), where

$$F(y, y') = y\sqrt{1 + y'^2},$$

we have

$$y\sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} = m ,$$

m being the constant of integration, a quantity which we shall consider later more in detail.

The above expression may be written

$$\frac{y\,(1 + y'^2 - y'^2)}{\sqrt{1 + y'^2}} = m ,$$

or

$$y = m\sqrt{1 + y'^2} . \tag{I}$$

From (I) follows directly that

$$y^2 - m^2 = m^2 \left[\frac{dy}{dx}\right]^2 ; \tag{II}$$

and (II) differentiated with respect to x, is

$$y = m^2 \frac{d^2 y}{dx^2} .$$

Two solutions of this differential equation are

$$y = e^{\frac{x}{m}} \quad \text{and} \quad y = e^{-\frac{x}{m}},$$

so that the general solution is

$$y = c_1 e^{\frac{x}{m}} + c_2 e^{-\frac{x}{m}}. \tag{III}$$

It appears that we have in this expression three arbitrary constants, m, c_1, and c_2; but from (II) we have, after substituting for y^2 and $\left[\frac{dy}{dx}\right]^2$ their values from (III),

$$m^2 = 4c_1c_2.$$

Hence, writing in (III),

$$c_1 = \tfrac{1}{2}m\, e^{-\frac{x_0'}{m}} \quad \text{and} \quad c_2 = \tfrac{1}{2}m\, e^{\frac{x_0'}{m}},$$

where x_0' is a constant, we have

$$y = \tfrac{1}{2}m\,(e^{(x-x_0')/m} + e^{-(x-x_0')/m}), \tag{(III$'$)}$$

which is the well known equation of the catenary.

The two constants x_0' and m are determined from the two conditions that the curve is to pass through the two fixed points P_0 and P_1.

REMARK. Equation (III$'$) takes the form

$$y = \tfrac{1}{2}m\,(e^t + e^{-t}),$$

when we write $x = x_0' + mt$.

7. *Properties of the catenary.*

Equation (II) above is

$$y^2 - m^2 = m^2y'^2,$$

or

$$\pm \sqrt{y^2 - m^2} = my'.$$

Therefore

$$dx = + \frac{mdy}{\sqrt{y^2 - m^2}}, \text{ or } - \frac{mdy}{\sqrt{y^2 - m^2}}. \tag{IV}$$

The integrals of these two equations may be written

$$\left.\begin{aligned} \frac{x - x_0'}{m} &= \log\left[\frac{y + \sqrt{y^2 - m^2}}{m}\right], \\ \text{and} \qquad -\frac{x - x_0}{m} &= \log\left[\frac{y - \sqrt{y^2 - m}}{m}\right]. \end{aligned}\right\} \tag{A}$$

Hence

$$\left.\begin{aligned} e^{\frac{x-x_0'}{m}} &= \frac{y+\sqrt{y^2-m^2}}{m}, \\ \text{and}\qquad e^{-\frac{x-x_0'}{m}} &= \frac{y-\sqrt{y^2-m^2}}{m}. \end{aligned}\right\} \tag{A'}$$

By the addition of equations (A′),

$$y = \tfrac{1}{2}\, m\,(e^{\frac{x-x_0'}{m}} + e^{-\frac{x-x_0'}{m}}). \tag{III'}$$

8. From (IV) we have

$$m\frac{dy}{dx} = \pm\sqrt{y^2-m^2},$$

and this from (III′) is

$$m\frac{dy}{dx} = \pm\sqrt{y^2-m^2} = \tfrac{1}{2}m\,(e^{\frac{x-x_0'}{m}} - e^{-\frac{x-x_0'}{m}}). \tag{V}$$

On the right hand side of this equation stands a one-valued function, but on the left hand side a two-valued function. Hence we must define the left hand expression so as to have a one-valued function.

If in the expression (V) we make $x > x_0'$, then

$$e^{\frac{x-x_0'}{m}} > e^{-\frac{x-x_0'}{m}},$$

and consequently $\sqrt{y^2-m^2}$ is positive. But when $x < x_0'$, then

$$e^{\frac{x-x_0'}{m}} < e^{-\frac{x-x_0'}{m}},$$

and then $\sqrt{y^2-m^2}$ is negative.

Hence from (V) there is only one root of the equation $dy/dx = 0$, and this is for the value $x = x_0'$.

The corresponding value of y is m. This value m is the smallest value that y can have; since $dy/dx = 0$ is the condition for maximum and minimum, and d^2y/dx^2 is positive, so that m is a minimum value of y; and since $\sqrt{y^2-m^2}$ is continuously positive or negative there is no maximum value of y.

Remark. The tangent to the curve at the point ($x = x_0'$, $y = m$) is parallel to the x-axis, because at this point, $dy/dx = 0$.

9. At every point of the curve we have

$$dy/dx = \tan\tau = \sqrt{y^2-m^2}/m\,.$$

Hence, to construct a tangent at any point of the catenary, for example at P, drop the perpendicular PQ, and describe the semi-circle on PQ as diameter.

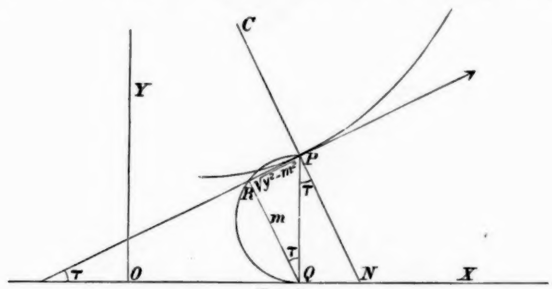

FIG. 1.

Then with radius equal to m, draw a circle from Q as centre, which cuts the semi-circle at R; join R and P. The line RP is the required tangent.

Again

$$ds^2 = dx^2 + dy^2 = \left[1 + \frac{y^2 - m^2}{m^2}\right] dx^2 = \frac{y^2 dx^2}{m^2};$$

consequently

$$ds = \frac{y\,dx}{m} = \tfrac{1}{2}\left(e^{(x-x_0')/m} + e^{-(x-x_0')/m}\right) dx;$$

and integrating,

$$s - s_0' = \tfrac{1}{2} m \left(e^{(x-x_0')/m} - e^{-(x-x_0')/m}\right) = \sqrt{y^2 - m^2},$$

where s_0' denotes that the arc is measured from the lowest point of the catenary.

The geometrical locus of R is a curve which cuts all the tangents to the catenary at right angles, and is therefore the *orthogonal trajectory* of this system of tangents. This trajectory has the remarkable property that the perpendiculars QR, etc., of length m, which are employed in the construction of the tangents to the catenary, are themselves tangent to the trajectory.

This trajectory possesses also the remarkable property that if we rotate it around the x-axis, the surface of rotation has a constant curvature.

Further, PN, the normal to the catenary, $= y \sec \tau = y^2/m$, and

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(\frac{ds}{dx}\right)^3}{\frac{d^2y}{dx^2}} = \frac{(y/m)^3}{y/m^2} = \frac{y^2}{m},$$

or

$$PN = PC \text{ (see Fig. 1)},$$

where PC is the length of the radius of curvature.

10. *The geometrical construction of the catenary.* Take an ordinate equal to $2m$. This determines the point P (see Fig. 2). With P as centre and radius

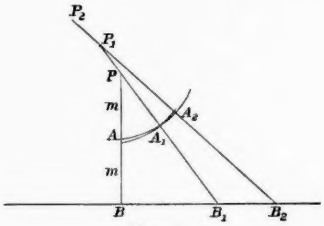

FIG. 2.

equal to m, describe a circle. This intersects PB at a point A, say. On the circumference of this circle, take a point A_1 very near A and draw the line PA_1B_1, and on this line extended take P_1 such that $P_1A_1 = A_1B_1$. With radius P_1A_1 draw another circle, and on this circle take a point A_2 very near the point A_1, and draw the line $P_1A_2B_2$. Take on this line extended the point P_2 so that $P_2A_2 = A_2B_2$, etc. The locus of the points A is the required catenary.

The accompanying figure shows approximately the relative positions of the catenary, its evolute, and the trajectory.

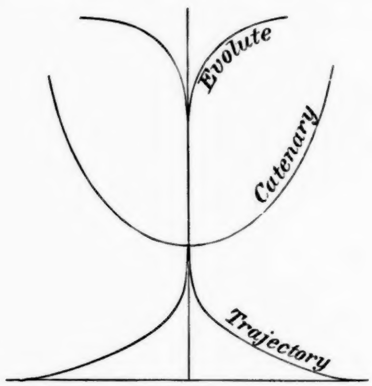

FIG. 3.

ERRATUM. In the last paragraph of Art. 2 (p. 82) the words positive and negative should be interchanged.

SOLUTIONS OF EXERCISES.

379

FIND the radius of a circle circumscribing the three tangent-circles of radii a, b, and c, respectively. [*F. P. Matz.*]

SOLUTION.

It may be easily shown, where α, β, γ, δ are sides of a quadrilateral, and λ and μ are diagonals, that the following relation exists:—

$$[4\alpha^2\lambda^2 - (\alpha^2 + \lambda^2 - \beta^2)^2]\,[4\lambda^2\delta^2 - (\lambda^2 + \delta^2 - \gamma^2)^2]$$
$$= \lambda^2\,[\alpha\delta\,(\alpha^2 + \lambda^2 - \beta^2)\,(\lambda^2 + \delta^2 - \gamma^2) - \lambda\,(\alpha^2 + \delta^2 - \mu^2)]^2\,.$$

From the geometry of the figure $\alpha = a + c$, $\beta = b + c$, $\gamma = R - b$, $\delta = R - a$, $\lambda = a + b$, $\mu = R - c$.

By substituting these values we obtain a quadratic in R. One value of R is the radius of circumscribed and the other the radius of inscribed circle. [*O. W. Anthony.*]

393

SHOW that if y be a quadratic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula, $\lambda y_1 + (1 - \lambda) y_2$, where y_1, y_2 correspond to the values

$$x_1 = \frac{h}{2} - h\sqrt{\frac{1}{12}\frac{1-\lambda}{\lambda}}\,, \quad x_2 = \frac{h}{2} + h\sqrt{\frac{1}{12}\frac{\lambda}{1-\lambda}}\,.$$

[*W. M. Thornton.*]

SOLUTION.

The solution is a transformation of the formula (14), ANNALS OF MATHEMATICS, Vol. IX, p. 6, wherein we have for the mean value required

$$(1 - \lambda)\,y_1 + \lambda y_2\,,$$

and

$$2\lambda = p + q\,, \tag{1}$$

with the condition

$$p^2 + pq + q^2 - \tfrac{3}{2}\,(p + q) = 0\,,$$

or

$$(p + q)^2 - \tfrac{3}{2}\,(p + q) = pq\,,$$

or

$$4\lambda^2 - 3\lambda = pq\,. \tag{2}$$

Combining (1) and (2), we obtain

$$p - q = \sqrt{12\lambda(1-\lambda)}.$$

Again

$$x_1 + p(x_2 - x_1) = 0,$$

$$x_1 + q(x_2 - x_1) = h;$$

whence

$$x_1 = \frac{h}{2} - \frac{h}{2}\frac{p+q}{p-q}$$

$$= \frac{h}{2} - h\sqrt{\frac{\lambda}{12(1-\lambda)}},$$

and

$$x_2 = \frac{h}{2} + h\sqrt{\frac{1-\lambda}{12\lambda}}.$$ [*W. H. Echols.*]

396

Show how to solve the simultaneous equations:

$$x = y\sin(z+\alpha) = y\sin z - a = y\sin(z+\beta) - b; \quad (1)$$

$$x(1-\sin y) = a, \quad x[1-\sin(y+\beta)] = b. \quad (2)$$

[*W. M. Thornton.*]

SOLUTION.

From (1)

$$y[\sin z - \sin(z+\alpha)] = a,$$

and

$$y[\sin z - \sin(z+\beta)] = a - b.$$

By division and substitution

$$\frac{\cos(z+\frac{1}{2}\alpha)}{\cos(z+\frac{1}{2}\beta)} = \frac{a\sin\frac{1}{2}\beta}{(a-b)\sin\frac{1}{2}\alpha}.$$

Expanding and dividing both terms of the first fraction by $\cos z$,

$$\frac{\cos\frac{1}{2}\alpha - \sin\frac{1}{2}\alpha\tan z}{\cos\frac{1}{2}\beta - \sin\frac{1}{2}\beta\tan z} = \frac{a\sin\frac{1}{2}\beta}{(a-b)\sin\frac{1}{2}\alpha},$$

from which

$$\tan z = \frac{a\sin\beta - (a-b)\sin\alpha}{a(1-\cos\beta) - (a-b)(1-\cos\alpha)}.$$

Having found z from above equation x and y are derived, in terms of z, from equation (1). Their values are

$$x = \frac{a\sin(z+\alpha)}{\sin z - \sin(z+\alpha)},$$

$$y = \frac{a}{\sin z - \sin(z+\alpha)}.$$ [*Marcus Baker.*]

397

Given two straight lines referred to rectangular coordinates; find geometrically an abscissa such that the sum of the squares of the two corresponding ordinates shall be a minimum. [*R. A. Harris.*]

SOLUTION.

Suppose the equation

$$y = mx$$

to represent the line having the greater inclination to the x-axis, and the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

to represent the other line. When

$$x = \frac{b^2 a}{a^2 m^2 + b^2},$$

the sum of the squares of the two y's becomes a minimum. The problem may now be stated: Given a, b, and m, to find x geometrically.

Let OC and AB denote the given lines; draw AC parallel to the y-axis, and lay off $AD = BO = b$. Bisect AO, thus determining E; then with E as center and CD as radius, describe the arc FGH. Draw BG parallel to the x-axis, take $GH = FG$, and draw the line EHI. Take $EI = EO = \frac{1}{2}a$, and project I upon the x-axis in J; then is OJ the required abscissa.

[*R. A. Harris.*]

398

If any curve, plane or gauche, be referred to a set of coordinates P, Q, which are so connected with a set of orthogonal and isothermal coordinates p, q that $P =$ function p, $Q =$ function q, then the angles made by this curve and the curves, $Q =$ constant, $P =$ constant are

$$\tan^{-1}\frac{dq}{dp}, \quad \tan^{-1}\frac{dp}{dq},$$

respectively. [*R. A. Harris.*]

SOLUTION.

Let (p, q) denote a point of the curve; then the distance to a neighboring point is $C\sqrt{dp^2 + dq^2}$ where C depends upon p, q, but is independent of dp, dq. The projections of this curve element upon $q =$ constant, $p =$ constant are Cdp, Cdq in length; hence

$$\tan^{-1}\frac{dq}{dp}, \quad \tan^{-1}\frac{dp}{dq}$$

are the angles made by the given curve and the curves $q =$ constant, $p =$ constant, which may be written $Q =$ constant, $P =$ constant.

[*R. A. Harris.*]

399

If y be any cubic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula

$$\tfrac{1}{6}(y_1 - 2y_2 + y_3)(1 + 2\sin^2\varphi) + \tfrac{1}{2}(y_3 - y_1)\sin\varphi + y_2,$$

where

$$x_2 = \frac{h}{\sqrt{2}}\sec\varphi\sin(45^\circ - \varphi)\,; \quad x_3 = x_2 + \tfrac{1}{2}h\sec\varphi\,; \quad x_1 = x_2 - \tfrac{1}{2}h\sec\varphi\,.$$

[*W. H. Echols.*]

SOLUTION.

We have, Annals of Mathematics, Vol. IX, p. 8, for the mean value of the cubic between 0, h,

$$\frac{1}{\rho+1}(\tfrac{1}{3}\rho S_2 - \tfrac{1}{2}S_1)\,y_1 - (\tfrac{1}{3}\rho S_2 + \overline{\rho - 1}\,\tfrac{1}{2}S_1 - 1)\,y_2 + \frac{\rho^2}{\rho+1}(\tfrac{1}{3}S_2 + \tfrac{1}{2}S_1)\,y_3$$
$$- (\tfrac{1}{4}\rho S_3 + \overline{\rho - 1}\,\tfrac{1}{3}S_2 - \tfrac{1}{2}S_1)\frac{h^3}{\rho(u-v)^3},$$

wherein

$$\rho = (x_2 - x_1)/(x_3 - x_2)$$

and

$$S_1 = u + v, \quad S_2 = u^2 + uv + v^2,$$

$$S_3 = u^3 + u^2v + uv^2 + v^3,$$

$$x_1 = -\frac{1+v}{u-v}h, \quad x_2 = -\frac{v}{u-v}h, \quad x_3 = \frac{1-\rho v}{u-v}\rho h.$$

Put

$$\tfrac{1}{4}\rho S_3 + \tfrac{1}{3}(\rho - 1)S_2 - \tfrac{1}{2}S_1 = 0,$$

and impose the condition that ρ shall be unity. Then follows the condition that u and v must satisfy $u^2 + v^2 = 2$.

Since, now, $S_2 = 1 + \tfrac{1}{2}S_1^2$, we have for the mean value

$$\tfrac{1}{6}(1 + \tfrac{1}{2}S_1^2)(y_1 - 2y_2 + y_3) + \tfrac{1}{4}S_1(y_3 - y_1) + y_2\,.$$

Put $u = \sqrt{2}\cos\theta$, $v = \sqrt{2}\sin\theta$. Then $uv = \sin 2\theta$, and $u + v = 2\sin(45^\circ + \theta)$; $u - v = 2\cos(45^\circ + \theta)$.

Let $\varphi = \theta + 45^\circ$ and substitute; then the mean value is

$$\tfrac{1}{6}(y_1 - 2y_2 + y_3)(1 + 2\sin^2\varphi) + \tfrac{1}{2}(y_3 - y_1)\sin\varphi + y_2,$$

where

$$x_2 = \frac{1}{\sqrt{2}} h \sec\varphi \sin(45° - \varphi),$$

$$x_3 = x_2 + \tfrac{1}{2}h \sec\varphi,$$

$$x_1 = x_2 - \tfrac{1}{2}h \sec\varphi.$$

[*W. H. Echols.*]

400

If y be any cubic function of x, its mean value in any interval $X_2 - X_1 = L$ can be expressed in an infinite number of ways in terms of only two ordinates by the formula

$$\frac{3\sin^2\varphi}{1+2\sin^2\varphi} y_3 + \left[1 - \frac{3\sin^2\varphi}{1+2\sin^2\varphi}\right] y_2 + \tfrac{1}{18} L^3 \frac{1-4\sin^2\varphi}{4\sin^2\varphi} \tan\varphi,$$

where

$$x_2 = X_1 + \frac{1}{\sqrt{2}} L \sec\varphi \sin(45° - \varphi), \quad x_3 = x_2 + \tfrac{1}{2} L \sec\varphi.$$

[*W. H. Echols.*]

SOLUTION.

In the general value for the mean value of the cubic given in the solution to Exercise 399 put

$$\rho = 3S_1/2S_2.$$

Then the general formula for the mean value is

$$\tfrac{1}{2}\rho S_1 y_3 + (1 - \tfrac{1}{2}\rho S_1) y_2 + \frac{2 - \rho^2(u^2+v^2)}{2\rho^2} \frac{S_1 L^3}{2(u-v)^3},$$

wherein

$$x_2 = X_1 - \frac{v}{u-v} L, \quad x_3 = x_2 + \frac{1}{u-v} L.$$

We may assign to u and v any values we choose, or we may choose x_2 and x_3 and get u and v, and thence the mean value in terms of y_3 and y_2. In particular let $u^2 + v^2 = 2$. Then, as before,

$$S_1 = u + v = 2\sin\varphi,$$

$$S_2 = u^2 + v^2 + uv = 2\cos\varphi,$$

$$uv = -\cos 2\varphi.$$

The mean value now becomes

$$\frac{3\sin^2\varphi}{1+2\sin^2\varphi} y_3 + \left[1 - \frac{3\sin^2\varphi}{1+2\sin^2\varphi}\right] y_2 + \tfrac{1}{18} L^3 \frac{1-4\sin^2\varphi}{4\sin^2\varphi} \tan\varphi,$$

and

$$x_2 = X_1 + \frac{1}{\sqrt{2}} L \sec \varphi \sin (45^\circ - \varphi)$$

$$x_3 = x_2 + \tfrac{1}{2} L \sin \varphi . \qquad \text{[} \textit{W. H. Echols.} \text{]}$$

402

Two circular arcs are tangent to each other and to the sides a and b of the triangle ABC at the points A and B. Show that the difference of curvature of the arcs is least when their common tangent makes with a and b the angles $\frac{1}{2}(3B - A)$ and $\frac{1}{2}(3A - B)$, respectively. [*W. H. Echols.*]

SOLUTION.

Let P be the point of tangency of the two arcs ;

θ, φ, the angles made by the common tangent with a and b ;

R_1, R_2, radii of the circles ;

r, the circum-radius of the triangle ABP.

Then

$$AP = 2R_1 \sin \tfrac{1}{2}\varphi = 2r \sin (A + B - \tfrac{1}{2}\theta) ,$$

$$BP = 2r_2 \sin \tfrac{1}{2}\theta = 2r \sin (A + B - \tfrac{1}{2}\varphi) ,$$

$$2r \sin \tfrac{1}{2}(A + B) = c ,$$

$$\theta + \varphi = A + B .$$

Accordingly,

$$\frac{1}{R_1} - \frac{1}{R_2} = \frac{2 \sin \frac{1}{2}(A + B)}{c} \left[\frac{\sin \frac{1}{2}\theta}{\sin (B - \frac{1}{2}\varphi)} - \frac{\sin \frac{1}{2}\varphi}{\sin (A - \frac{1}{2}\theta)} \right] .$$

The expression in brackets reduces to

$$\frac{\cos A - \cos B}{\cos \frac{1}{2}(A + B) - \cos \frac{1}{2}(A - 3B + 2\varphi)} ,$$

which is a minimum when

$$A - 3B + 2\varphi = 0 ,$$

i. e. when

$$\varphi = \tfrac{1}{2}(3B - A)$$

and

$$\theta = \tfrac{1}{2}(3A - B) . \qquad \text{[} \textit{Geo. R. Dean.} \text{]}$$

ON THE SOLUTION OF A CERTAIN DIFFERENTIAL EQUATION WHICH PRESENTS ITSELF IN LAPLACE'S KINETIC THEORY OF TIDES.

By Mr. George Herbert Ling, New York, N. Y.

TABLE OF CONTENTS.

I.

Introductory.

1. *Objects of the paper.* In his discussion of the kinetic theory of tides, Laplace found that the function expressing the height of the tide at a given point due to the attraction of the disturbing body satisfied a certain differential equation. Finding himself unable to obtain the general solution of the differential equation, he applied himself to the discussion of several particular cases which arise when certain assumptions are made regarding the physical constitution of the ocean. One of the cases he treated was that of the semidiurnal tide when the depth of the ocean is supposed to be constant. In the course of his treatment of this case certain considerations enter which have given rise to much discussion. It is proposed to devote some attention to this case, and it is hoped to extend the treatment of this case so as to include some phases of it not previously treated. While it is generally conceded that the facts in regard to the disputed point referred to, have been made evident, yet the methods of placing those facts in evidence have been called into question by several writers on the subject, and do not appear to be the most satisfactory ones that are available.

II.

Historical Sketch.

2. *Origin of the problem.* As just mentioned, the subject to be discussed was first treated by Laplace. His kinetic theory of tides is set forth in the Mécanique Céleste, and the part with which we are concerned is to be found in Livre IV of that work, his solution of the differential equation being given in Article 10. Considerable time had elapsed between his first discussion of the subject and the publication of his great work. The earliest presentation of his treatment of the subject was contained in a memoir* presented to the Académie des Sciences, and contained in Tome IX of the Œuvres de Laplace. He has sought a solution of the equation in the form of a series of positive entire powers, and has made use of a certain infinite continued fraction in the evaluation of one of the coefficients of the series. The correctness of the value found by his method has been questioned. As the solution in the series form was made the basis of his calculations, it was of great importance that no mistake should be made in the determination of the coefficients, and more especially in the determination of those occurring early in the series.

* Recherches sur plusieurs points du Systeme du Monde. Memoires de l'Academie royale de Paris, année 1775–6.

3. *Previous contributions to the subject.* In his early memoir Laplace has gone somewhat more into detail, and the method by which he determined the value of the coefficient is clearly shown. In the later work he has omitted a great part of the explanation, and has contented himself with expressing the quantity in the form of the continued fraction to which reference has been made. The later presentation of the subject has been the more accessible of the two, and on it all the later writers appear to have based their remarks concerning Laplace's method, while the original presentation has been overlooked. Attention has been called to it by Prof. Lamb in his recent work on Hydrodynamics, and to him seems to be due the rediscovery, so to speak, of the memoir. Laplace's evaluation of the coefficient was objected to by Sir G. B. Airy,* and later the same objection was made by Mr. William Ferrel.† A defence of Laplace was made by Lord Kelvin in the Philosophical Magazine for September, 1875. The October number of the same journal for 1875 contains a note written by Airy in which he reaffirms his objections to Laplace's result, and appears not to regard Kelvin's reasoning as convincing. The number of this journal for March, 1876, contains a reply by Ferrel to the arguments of Lord Kelvin. Prof. G. H. Darwin in the Encyclopedia Britannica‡ gives in more detail Lord Kelvin's argument. His treatment of the subject may also be found in Basset's Hydrodynamics, Vol. II, and Basset has briefly referred to the subject in a foot note.§ The latest contributions to the subject are believed to be the two papers by Ferrel which appear in Volumes 9 and 10 of Gould's Astronomical Journal. The latter of the two papers may also be found in the collection of papers ‖ on the "Mechanics of the Atmosphere" edited by Prof. Cleveland Abbe. Reference may also be made to Professor Lamb's Hydrodynamics, in which attention is called to Laplace's original memoir.

4. *Synopsis of previous contributions.* Before treating the problem analytically it will be useful to sketch the arguments of Laplace and those who afterwards treated the subject. Laplace, assuming that the solution of the equation could be expressed by means of a Taylor's series, substituted such a series with undetermined coefficients in the differential equation, and was able to determine all the coefficients of the series in terms of one of them, which remained arbitrary. He had previously argued that it was not necessary to obtain the general solution of the equation, since, as he affirmed, the arbitrary constants would be determined by the initial conditions of the water and would introduce effects dependent on this initial condition, which effects ought to be

* Article, "Tides and Waves," Encyclopedia Metropolitana.

† "Tidal Researches," Appendix to United States Coast and Geodetic Survey Report for 1874.

‡ Article, "Tides," Encyclopedia Britannica.

§ Basset, Vol. II, p. 218.

‖ No. 843, Smithsonian Miscellaneous Contributions.

disregarded, since in the case of the sea they would long ago have been overcome by friction. Considering, then, that any particular integral was sufficient, he proceeded to choose the most satisfactory value of the coefficient. His method of deciding the proper value of the coefficient will be given in Section III. It enabled him to satisfy himself that a comparatively few terms of his series would give the result with a very small error. From the form in which the result is set forth in the Mécanique Céleste it appears, however, that he made the assumption that the ratio of any coefficient in his series to the preceding one becomes ultimately smaller than any assignable quantity. Moreover Laplace's argument regarding the sufficiency of any particular solution did not occur immediately in connection with the treatment of this particular case, and it therefore appeared that he offered no justification for the assumption. Airy objected to the assumption on the ground that it was unnecessary and unduly specialized the solution. He added that, if the sea were bounded by a parallel of latitude instead of covering the whole earth, then the arbitrary constant could be determined from the corresponding boundary condition. Ferrel agreed with Airy, and regarding the constant as being entirely at his disposal, based his calculations on the series resulting from assigning to it the value zero. Kelvin in his reply to these arguments quoted Laplace's reasoning regarding the sufficiency of a particular solution, but pointed out that this reasoning was not correct. Proceeding, he contended that, as demanded by Airy, there was a certain physical condition to be satisfied, and that this condition was sufficient to justify Laplace's result. He also pointed out that the general solution of the equation should contain two arbitrary constants, and that a further boundary condition would be necessary for the determination of the second of these constants. He showed that, since the oscillations of the water which are taken account of in the differential equation have a perfectly definite period depending on the period of the disturbing body, the original state of motion could not be taken account of in the solution, for, except for special depths of water, the period of the latter oscillations would be different from that of the former. Airy and Ferrel, however, did not admit the force of the reasoning by means of which Kelvin justified Laplace's result, and Ferrel's later papers are devoted to an attempt to show that the determination of the value of the constant is unnecessary. While it would seem that the constant was correctly determined by Laplace, it appears to the author that the analytical proof of this fact indicated by Lord Kelvin, and given in greater detail by Prof. Darwin, is not complete. It seems, too, to be desirable to obtain the general solution of the differential equation, and to follow up the suggestions of Lord Kelvin regarding the application of this solution to the more general case and some special cases.

III.

Laplace's Solution.

5. *General outline.* Apart from physical considerations, the arguments made by Laplace from analysis do not appear to offer a very good reason for his evaluation of the coefficient. In order to show this, it will be necessary to give Laplace's discussion as it appeared in his original memoir. The form of the equation as treated by Darwin differs slightly from that in which Laplace used it, but no essential difference is introduced by the change, and the same difficulties arise in both cases. As Darwin's form of the equation is doubtless that in which the equation will henceforth be studied, it has been adopted here. The equation may then be written

$$x^2(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} - (8 - 2x^2 - \beta x^4)\,u + E\beta x^6 = 0\,. \tag{1}$$

It is to be noted that x is the sine of the polar distance of a particle of water, u the difference between the tide height in the dynamical theory and the tide height in the equilibrium theory. Assuming as the solution of (1) a Taylor's series containing only even powers and with undetermined coefficients, Laplace found that the coefficient of x^4 remained undetermined. He next proceeded as follows: He assumed as an integral a sum of a finite number of powers, and found that by adding a certain term to the left member of (1) this equation could be modified so as to have as a solution the assumed function. By studying the effect of increasing the number of terms in this function, he came to the conclusion that a very small error would be made in assuming as a solution, such a function with a large number of terms.

6. *Application.* To apply this treatment to the equation (1), assume that

$$u = A_0 + (A_1 - E)\,x^2 + \sum_{2}^{r+1}{}_k\, A_k x^{2k}\,. \tag{2}$$

If this function satisfies equation (1), the following relations must be satisfied by the coefficients:

$$A_0 = 0\,, \qquad \text{(a)}$$

$$A_1 - E = 0\,, \qquad \text{(b)}$$

$$A_2 = A_2\,, \qquad \text{(c)}$$

$$16A_3 - 10A_2 + \beta A_1 = 0\,, \qquad \text{(d)}$$

$$2A_{k+1}[2(k-1)^2 + 6(k-1)] - 2A_k[2(k-1)^2 + 3(k-1)] + \beta A_{k-1} = 0\,, \qquad \text{(e)}$$

$$(k = 3, 4, 5, \ldots, r.)$$

$$-\,2A_{r+1}(2r^2 + 3r) + \beta A_r = 0\,, \qquad \text{(f)}$$

$$+\,\beta A_{r+1} = 0\,. \qquad \text{(g)}$$

There are, since (c) is an identity, $r+3$ linear equations to be satisfied by $r+2$ unknown quantities. It is easy to see that all cannot be satisfied ; for, starting from (g) and working back, there result zero values for all the A's and this disagrees with (b). If, however, one of the equations be rejected, the remaining $r+2$ relations are sufficient to determine the values of the A's. Suppose (g) to be rejected. Following Laplace's method let the following abbreviations be made :

$$\mu = \tfrac{1}{2}\beta\,,$$

$$\mu_r = 2r^2 + 3r\,,$$

$$\mu_{r-1} = 2(r-1)^2 + 3(r-1) - [(r-1)^2 + 3(r-1)]\,\mu/\mu_r\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mu_{r-k} = 2(r-k)^2 + 3(r-k) - [(r-k)^2 + 3(r-k)]\,\mu/\mu_{r-k+1}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Then it follows that

$$A_{r+1} = \frac{\mu}{\mu_r} A_r\,,$$

$$A_r = \frac{\mu}{\mu_{r-1}} A_{r-1}\,,$$

$$A_{r-1} = \frac{\mu}{\mu_{r-2}} A_{r-2}\,,$$

$$\cdots\cdots\cdots\cdots$$

$$A_2 = \frac{\mu}{\mu_1} A_1 = \frac{\mu}{\mu_1} E\,;$$

whence

$$A_3 = \frac{\mu^2}{\mu_1\mu_2} E\,,$$

$$A_4 = \frac{\mu^3}{\mu_1\mu_2\mu_3} E\,,$$

$$\cdots\cdots\cdots\cdots$$

$$A_{r+1} = \frac{\mu^r}{\mu_1\mu_2\mu_3\ldots\mu_r} E\,.$$

These values of $A_1, A_2, A_3, \ldots, A_{r+1}$ satisfy the equations (a), (b), (c), (d), (e) (f), but equation (g) is not satisfied, and the function (2) is therefore not a solution of (1). If, however, to the left member of (1) be added the quantity $-\beta A_{r+1}x^{2r+6}$, equation (g) becomes the identity

$$-\beta A_{r+1} + \beta A_{r+1} = 0\,,$$

so that the expression (2) is a solution of equation (1) thus modified. The new equation can be written

$$x^2(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} - u(8 - 2x^2 - \beta x^4) + E\beta x^6 - \frac{E\beta^r}{\mu_1\mu_2\ldots\mu_r}x^{2r+6} = 0. \quad (3)$$

Laplace then argued that if the corrective term were very small, only a small error would be made if (1) were replaced by (3). He then proceeded to show that by taking r large enough the corrective term could be made to decrease indefinitely.

7. *Objections to method.* In order that the discussion just given may justify the choice of the value of A_2, it should put in evidence some property possessed by the series when Laplace's value is given to A_2, and not possessed by it under other circumstances. All that is attempted in the preceding process is to show that the error made in assuming as the integral a finite number of terms belonging to the infinite series can be made less than any assignable quantity. But this is true of any series which is an integral of the equation and whose region of convergence is large enough to suit the conditions of the problem, and it will appear that no matter what value be given to A_2 the region of convergence of the resulting series is still large enough to suit the purpose. Moreover, it is not definitely proven that a vanishing correction to the equation necessarily indicates a corresponding vanishing of the error due to the assumption of a finite number of terms instead of an infinite number.

8. *Relation between correction to equation and error in series.* The relation between the correction to the equation and the error in the value of the dependent variable can be shown perhaps more clearly in the following manner, assuming certain properties of the series used which will be deduced later: The differential equation (1) can be regarded as a linear relation connecting the quantities u, $\frac{du}{dx}$, and $\frac{d^2u}{dx^2}$ in which the coefficients are rational entire functions of x. The function u is to be expressed by means of an infinite series. This series will have a certain circle of convergence. For all points *within* this circle the corresponding series for $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ will also converge. Suppose the circumference of this circle of convergence lies entirely outside of the boundary of the region in which the independent variable is to vary. Then when u, $\frac{du}{dx}$, and $\frac{d^2u}{dx^2}$ are each replaced by the finite number of terms from the series expressing their values, certain errors will be made in the case of each. These errors will each become less than any assignable quantity if the number of terms be sufficiently increased. Let (1) then be written

$$\alpha\frac{d^2u}{dx^2} + \beta_1\frac{du}{dx} + \gamma u = \delta,$$

where α, β_1, γ, and δ are finite for all values to be considered. Let the true values of $\frac{d^2u}{dx^2}$, $\frac{du}{dx}$, and u when found from the infinite series be A, B, and C; and let ε_1, ε_2, and ε_3 be the corresponding errors made in taking the finite number of terms for each of the three quantities. Then

$$\alpha A + \beta_1 B + \gamma C = \delta\,, \tag{h}$$

and

$$\alpha(A + \varepsilon_1) + \beta_1(B + \varepsilon_2) + \gamma(C + \varepsilon_3) \gtrless \delta\,. \tag{k}$$

Then the correction to be added to the left member of (k) to make it an equality is $-(\alpha\varepsilon_1 + \beta_1\varepsilon_2 + \gamma\varepsilon_3)$. Under the circumstances assumed above, in reference to the region of convergence, this correction will become indefinitely small when the number of the terms is sufficiently increased. But other cases may occur. Consider the case of the infinite series in which A_2 is given any other value than that assigned by Laplace. It will afterwards appear that this series converges for all values of x within the unit circle, and also for $x = \pm 1$. The series for $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$, however, are convergent only for points within the unit circle. It is clear then that, for all values of x less than unity, ε_1, ε_2, and ε_3 can be made less than any assignable quantity, and that therefore the same is true for $\alpha\varepsilon_1, + \beta_1\varepsilon_2, + \gamma\varepsilon_3$.

But x must be considered for all values up to and including unity, and it does not appear that, as x approaches unity, the correction must necessarily indefinitely diminish, since ε_3 is the only one of the three quantities which indefinitely diminishes. Moreover the series is a satisfactory solution if only ε_3 can be made indefinitely small for $x = 1$. On the other hand, it would also appear that the correction might be evanescent when ε_3 is not so.

9. *Assumption equivalent to Laplace's assumption.* The same set of equations for the determination of the coefficients will be obtained if it be assumed that

$$\underset{r=\infty}{\mathfrak{L}} \frac{A_{r+1}}{A_r} = 0\,.$$

For then it would be proper to assume that the equations connecting the coefficients eventually took the same form as (f). For all other series in which the relation just written is not true the equation (f) is not satisfied when a finite number of terms is taken, but has to be corrected by the addition of a term. From this point of view then Laplace's process would seem to necessarily lead to a series convergent over the entire plane.

IV.

The Solution of the Differential Equation.

10. *Character of the integral.* Laplace's solution having been considered, the general integral of the equation,

$$x^2(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} - u(8-2x^2-\beta x^4) = -\beta E x^6, \tag{1}$$

may now be sought. In this equation it must be remembered that $x = \sin\theta$ where θ is a polar distance. It is necessary first to solve the auxiliary equation,

$$x^2(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} - u(8-2x^2-\beta x^4) = 0. \tag{7}$$

The following general theorems will be useful:

Theorem I. *In order that the equation*

$$\frac{d^n u}{dx^n} + p_1\frac{d^{n-1}u}{dx^{n-1}} + p_2\frac{d^{n-2}u}{dx^{n-2}} + p_3\frac{d^{n-3}u}{dx^{n-3}} + \dots + p_{n-1}\frac{du}{dx} + p_n u = 0 \tag{8}$$

shall have n independent integrals of the form,

$$u = x^r(P_{n-1}\log^{n-1}x + P_{n-2}\log^{n-2}x + P_{n-3}\log^{n-3}x + \dots + P_1\log x + P_0),$$

where $P_0, P_1, P_2, P_3, \dots, P_{n-1}$ are expressible in the neighborhood of $x = 0$, in series of positive and negative powers of x, the number of negative powers in each being finite, it is necessary and sufficient that for each of the coefficients of the equation, such as p_i, the point $x = 0$ shall be an ordinary point, or a pole whose order of multiplicity does not exceed i.

These integrals will constitute one or more groups of the form

$$u_1 = x^r M_1,$$
$$u_2 = x^r(M_2\log x + N_2),$$
$$u_3 = x^r(M_3\log^2 x + N_3\log x + R_3).$$
$$\dots\dots\dots\dots\dots\dots$$
$$u_k = x^r(M_k\log^{k-1}x + N_k\log^{k-2}x + \dots),$$

where $M_1, M_2, M_3, \dots, M_k$ differ only by constant factors.

Theorem II. *The integral of the equation* (8) *will be continuous and monogenic for all values of x for which the coefficients $p_1, p_2, p_3, \dots, p_n$ are continuous and monogenic, and it can possess no critical points which are not*

also critical points of one or more of the coefficients. It may not have critical points at all the critical points of the coefficients.

The proofs of these theorems may be found in Jordan's *Cours d'Analyse*.*

If, then, equation (7) be put in the form (8) and these theorems applied, it is clear that,

(1) Equation (7) has one integral which can be expressed in the form of a power series ψ, and a second integral which can be expressed in the form $\psi_2 + \psi_1 \log x$, where ψ_2 and ψ_1 are expressible as power series and $\frac{\psi_1}{\psi} =$ a constant c (when $c = 0$, the second integral is also expressible as a power series);

(2) The general integral of (7) can have no critical points except at 0, ± 1, $\pm \infty$.

(3) If x be replaced by $-x$, the equation remains unaltered; whence it follows that the function has the same character at $x = -a$ as at $x = +a$, and, in particular, if the general integral of (7) has a critical point of any sort at $x = +1$ it has a critical point of exactly the same character at $x = -1$.

It follows, too, that if U is any particular integral of (1), the complete integral is

$$u = a_1\psi + a_2(\psi_2 + c\psi \log x) + U,$$

or

$$u = a_1\psi + a_2\psi_2 + U$$

when $c = 0$.

11. *Deduction of the complementary function.* It is correct, then, to assume as one integral,

$$u = \sum_0^\infty {}_r A_r x^{m+rs},$$

s being a positive integer. Then

$$\frac{du}{dx} = \sum_0^\infty {}_r (m + rs) A_r x^{m-1+rs},$$

$$\frac{d^2u}{dx^2} = \sum_0^\infty {}_r (m + rs)(m - 1 + rs) A_r x^{m-2+rs},$$

The result of substituting in (7) is

$$\sum_0^\infty {}_r (m + rs)(m - 1 + rs) A_r x^{m+rs} - \sum_0^\infty {}_r (m + rs)(m - 1 + rs) A_r x^{m+2+rs}$$

$$- \sum_0^\infty {}_r (m + rs) A_r x^{m+rs} - \sum_0^\infty {}_r 8 A_r x^{m+rs}$$

$$+ \sum_0^\infty {}_r 2 A_r x^{m+2+rs} + \sum_0^\infty {}_r \beta A_r x^{m+4+rs} = 0, \qquad (9)$$

* Vol. III, Arts. 146, 92, 118.

identically. The coefficient of x^m, the lowest power of x, must vanish ; whence, since $A_0 \gtrless 0$,

$$m^2 - 2m - 8 = 0\,, \text{ and } m = 4\,,\ -2\,.$$

The value of s must be 2, for if $s \gtrless 2$ the coefficient of x^{m+2} does not vanish. The result of substituting $s = 2$, $m = 4$ in (9) is

$$\begin{aligned}&\sum_0^\infty{}_r\,(4+2r)(3+2r)A_r x^{2r+4} - \sum_0^\infty{}_r\,(4+2r)(3+2r)A_r x^{2r+6}\\ &-\sum_0^\infty{}_r\,(4+2r)A_r x^{2r+4} - \sum_0^\infty{}_r\,8A_r x^{2r+4}\\ &+\sum_0^\infty{}_r\,2A_r x^{2r+6} + \sum_0^\infty{}_r\,\beta A_r x^{2r+8} = 0\,. \qquad (10)\end{aligned}$$

From (10) the equations for the determination of the coefficients are,

$$16A_1 - 10A_0 = 0\,, \qquad (11)$$

$$2k\,(2k+6)\,A_k - 2k\,(2k+3)\,A_{k-1} + \beta A_{k-2} = 0\,, \qquad (12)$$

$$(k = 2, 3, 4, 5, \ldots)\,.$$

Thus each coefficient is a multiple of A_0 and one integral can be written

$$\begin{aligned}u_1 &= x^4 + \frac{A_1}{A_0}x^6 + \frac{A_2}{A_0}x^8 + \frac{A_3}{A_0}x^{10} + \ldots\\ &= x^4 + C_1x^6 + C_2x^8 + C_3x^{10} + \ldots\,.\end{aligned} \qquad (13)$$

It is easy to verify that $K_4\,u_1$ is the series written down by Airy in his article in the Philosophical Magazine as the correction to Laplace's series, K_4 being an arbitrary constant. On applying to the equation the theory set forth by Heffter,* it appears that there is no series corresponding to the root $m = -2$. The second integral is then obtained in the form

$$u_2 = \psi_2 + \psi_1 \log x = \psi_2 + A'\,u_1 \log x\,.$$

From this

$$\frac{du_2}{dx} = \frac{d\psi_2}{dx} + A' \log x\,\frac{du_1}{dx} + \frac{A'}{x}\,u_1\,,$$

$$\frac{d^2u_2}{dx^2} = \frac{d^2\psi_2}{dx^2} + \frac{2A'}{x}\frac{du_1}{dx} - \frac{A'}{x^2}\,u_1 + A' \log x\,\frac{d^2u_1}{dx^2}\,.$$

* Lineare Differential gleichungen, § 16.

These values being substituted in (7), there results

$$x^2(1-x^2)\frac{d^2\psi_2}{dx^2} - x\frac{d\psi_2}{dx} - \psi_2(8-2x^2-\beta x^4)$$

$$- A'u_1(2-x^2) + A'\,.\,2x(1-x^2)\frac{du_1}{dx}$$

$$+ A'\log x\left[x^2(1-x^2)\frac{d^2u_1}{dx^2} - x\frac{du_1}{dx} - u_1(8-2x^2-\beta x^4)\right] = 0\,, \qquad (15)$$

identically. If the substitution $\psi_2 = A'\psi_3$ be made and it be borne in mind that u_1 is a solution of (7), (15) reduces to

$$x^2(1-x^2)\frac{d^2\psi_3}{dx^2} - x\frac{d\psi_3}{dx} - \psi_3(8-2x^2-\beta x^4)$$

$$- (2-x^2)\,u_1 + 2x(1-x^2)\frac{du_1}{dx} = 0\,. \qquad (16)$$

If now it be assumed that

$$\psi_3 = \sum_0^\infty{}_r\, B_r x^{m+rs}\,, \qquad (17)$$

and the substitution be made in (16), the equation for the determination of m is, as before,

$$m^2 - 2m - 8 = 0\,, \text{ whence } m = 4,\ -2\,.$$

The theory shows that $m = -2$ is the root to be taken, and that, as before, $s = 2$. The substitution in (16) gives

$$\sum_0^\infty{}_r\,(2r-2)(2r-3)B_r x^{2r-2} - \sum_0^\infty{}_r\,(2r-2)(2r-3)B_r x^{2r}$$

$$- \sum_0^\infty{}_r\,(2r-2)B_r x^{2r-2} - \sum_0^\infty{}_r\,8B_r x^{2r-2} + \sum_0^\infty{}_r\,2B_r x^{2r}$$

$$+ \sum_0^\infty{}_r\,\beta B_r x^{2r+2} - \sum_0^\infty{}_r\,2C_r x^{2r+4} + \sum_0^\infty{}_r\,C_r x^{2r+6}$$

$$+ \sum_0^\infty{}_r\,4(r+2)C_r x^{2r+4} - \sum_0^\infty{}_r\,4(r+2)C_r x^{2r+6} = 0\,, \qquad (18)$$

in which the C's are the coefficients in the series u_1.

From (19) are derived the relations

$$
\begin{aligned}
0 &= 8B_1 + 4B_0\,, \\
0 &= 8B_2 - 2B_1 - \beta B_0\,, \\
6 &= \qquad\qquad - \beta B_1\,, \\
7 - 10C_1 &= 16B_4 - 10B_3 + \beta B_2\,,
\end{aligned}
\tag{19}
$$

$$
(4k-5)\,C_{k-3} - (4k-2)\,C_{k-2} = (2k-4)(2k+2)\,B_{k+1} - (2k-4)(2k-1)\,B_k + \beta B_{k-1}\,.
$$

$$(k = 4, 5, 6, \ldots)$$

These equations determine all the coefficients in terms of B_3, the coefficient of x^4, which remains arbitrary.

Since u_1 is a solution of (7), it is clear that, if φ is a solution of (16), so also is $\varphi + A'_3u_1$, A'_3 being any constant. Moreover u_1 starts with the fourth power of x. Then if any value be assigned to B_3 and the resulting value of ψ_3 be denoted by φ we can write

$$\psi_3 = \varphi + A'_3u_1\,.$$

φ is a series of ascending entire powers starting from B_0x^{-2} in which every coefficient is known. B_3 may, if it is desired, be taken to be zero. The complete complementary function of (17) then is

$$
\begin{aligned}
u &= A'_1(\psi_2 + \psi_1 \log x) + A'_2u_1 \\
&= A'_1(\psi_2 + A'u_1 \log x) + A'_2u_1 \\
&= A'A'_1(\psi_3 + u_1 \log x) + A'_2u_1 \\
&= A(\varphi + u_1 \log x) + Bu_1\,.
\end{aligned}
\tag{20}
$$

12. *The particular integral.* It remains to determine a particular integral of the complete equation (1). The character of u_1 and of the absolute term of (1) makes clear the existence of a particular integral expressible in the form of a series of positive entire powers. Assume then

$$U = A_0 + (A_1 - E)\,x^2 + \sum_{r=2}^{\infty} A_r x^{2r}\,.$$

After substitution in (1) there result the relations

$$
\begin{aligned}
A_0 &= 0\,, \\
A_1 - E &= 0\,, \\
A_2 &= A_2\,,
\end{aligned}
$$

$$2A_{k+1}\{2(k-1)^2 + 6(k-1)\} - 2A_k\{2(k-1)^2 + 3(k-1)\} + \beta A_{k-1} = 0\,.$$

$$(k = 2, 3, 4, \ldots)$$

All the coefficients are given in terms of E and A_2, the latter being arbitrary. Since only a particular integral is required any value may be given to A_2. In proceeding to a choice of a value it is interesting to follow the method by which Laplace's continued fraction is obtained. From the relations written just above, it can easily be deduced that

$$\frac{A_{k+1}}{A_k} = \frac{{}_i\beta}{2\,(2k^2 + 3k) - 2\,(2k^2 + 6k)\, A_{k+2}/A_{k+1}} \,. \quad (k = 2, 3, 4, \ldots)$$

$$\therefore \quad \frac{A_2}{A_1} = \frac{{}_i\beta}{2(2\,.\,1^2 + 3\,.\,1) -}\;\frac{(2\,.\,1^2 + 6\,.\,1)\,{}_i\beta}{2\,(2\,.\,2^2 + 3\,.\,2) -}\;\frac{(2\,.\,2^2 + 6\,.\,2)\,{}_i\beta}{2\,(2\,.\,3^2 + 3\,.\,3) -}\cdots$$

$$\frac{\{2\,(n-1)^2 + 6\,(n-1)\}\,{}_i\beta}{2\,[2\,(n-1)^2 + 3\,(n-1)] - 2\,[(2n^2 + 3n) - (2n^2 + 6n)]\, A_{n+2}/A_{n+1}} \,,$$

and $A_1 = E$.

The assumption,

$$\mathop{\mathcal{L}}_{n=\infty} \frac{A_{n+2}}{A_{n+1}} = 0 \,, \tag{21}$$

gives the value of A_2 in the form of an infinite continued fraction. It is permissible, if convenient, since any value of A_2 may be taken. It was made by Laplace in the Mécanique Céleste apparently without justification; but, as has been seen, Laplace believed in the sufficiency of a particular solution and considered the resulting series as a satisfactory solution without the addition of the complementary function. The assumption (21) so affects the coefficients that the series converges for all finite values of x. It is not necessary that the particular integral should converge for points outside the unit circle. It is convenient from a mathematical point of view, however, to choose this series as the particular integral, for, if it were necessary to study the function for points outside the unit circle, it would be sufficient to obtain the complementary function in the form of a Laurent's series while the series just found would serve again as a particular integral. The assumption (21) is seen to be equivalent to that involved in Laplace's original process. From the reasoning of this section it is clear, too, that when

$$\mathop{\mathcal{L}}_{n=\infty} \frac{A_{n+1}}{A_n} \gtreqless 0 \,,$$

then

$$\mathop{\mathcal{L}}_{n=\infty} \frac{A_{n+1}}{A_n} = 1 \,.$$

From this point of view, too, it is clear that the series under consideration converges at least for every point within the unit circle, and that if it converges for a greater circle of convergence it converges for every finite value of x. For the sake of definiteness Laplace's value of A_2 will be denoted by L and the series which furnishes the particular integral of (1) will be denoted by $\mathfrak{P}$.

13. *Properties of the complete integral.* The complete integral of the equation (1) can then be expressed for points within the domain of the origin by

$$u = A\,(\varphi + u_1 \log x) + Bu_1 + \mathfrak{P}\,. \tag{22}$$

(1) Convergence:

The most general integral in the form of a positive power series can be written

$$u = B\,u_1 + \mathfrak{P}\,.$$

The relations among the coefficients of such a series show that, unless B is zero, the circle of convergence is of unit radius; and when B is zero, the circle of convergence has an indefinitely great radius.* It follows, then, that u_1 converges only for points within or on the circle of unit radius. Again $\varphi_3 = \varphi + A_2 u_1$ does not converge for points outside the unit circle. Then φ converges for points within the unit circle. It is conceivable that B_3 may have been chosen so that φ shall converge all over the finite part of the plane.

(2) If $A = B = 0$, the function has no critical point except at infinity. If $A = 0$, $B \gtrless 0$, the function has critical points at ± 1, $\pm \infty$. If $B = 0$, $A \gtrless 0$, the function has a critical point at 0, $\pm \infty$, and (except for one particular choice of B_3) at ± 1. In addition to the singularity of φ at $x = 0$ integrals of this class and integrals of the general class have a singularity at $x = 0$ due to the singularity of $\log x$ at that point, and are, in addition, many valued at any point owing to the properties of $\log x$. This indeterminateness will be removed if it is assumed that for positive real values of x the result shall be real. Again, when $B = 0$,

$$\frac{du}{dx} = A\left[\frac{d\varphi}{dx} + \log x\,\frac{du_1}{dx} + \frac{1}{x}u_1\right] + \frac{d\mathfrak{P}}{dx}.$$

But

$$\frac{du}{d\theta} = \frac{du}{dx}\sqrt{1 - x^2}\,,$$

$$\mathfrak{L}\left[\frac{du}{d\theta}\right]_{\theta=\pi/2} = \mathfrak{L}_{x=1}\left[\frac{du}{dx}\sqrt{1 - x^2}\right]$$

$$= A\,\mathfrak{L}_{x=1}\left[\sqrt{1 - x^2}\,\frac{d\varphi}{dx}\right] + A\,\mathfrak{L}_{x=1}\left[(\log x)\sqrt{1 - x^2}\,\frac{du_1}{dx}\right].$$

* The detailed proof of this fact is quoted in Section VI.

It will afterwards be shown that for the complete integral (22)

$$\left[\frac{du}{d\theta}\right]_{\theta=\pi/2} = \text{a finite quantity,}$$

and that

$$\left[\frac{du_1}{d\theta}\right]_{\theta=\pi/2} = \text{a finite quantity.}$$

$\therefore$ for all integrals of the form

$$u = A\,(\varphi + u_1 \log x) + \mathfrak{L}\,,$$

$$\left[\frac{du}{d\theta}\right]_{\theta=\pi/2} = \text{a finite quantity or zero.}$$

V.

The Determination of the Constants for Laplace's Case.

14. *The physical conditions.* It having been agreed that the constants shall be determined to suit the boundary conditions, the case discussed by Laplace, where the whole earth is covered with water, may now be treated. The expression for u, as given in (22), has an infinite value when $x = 0$, unless $A = 0$. Since there cannot be a tide of infinite depth at the pole it is necessary to make $A = 0$. The remaining expression is

$$u = Bu_1 + \mathfrak{L}\,.$$

Airy and Ferrel contended that this was the exact expression for u, and that B could be given any value. Ferrel determined it by the condition

$$B + L = 0\,. \tag{23}$$

Lord Kelvin pointed out that owing to the symmetry of the disturbance in the two hemispheres the meridional displacement of water should vanish at the equator. The expression for the meridional displacement is the product of two terms of which one involves the latitude and the other does not. The factor involving the latitude is,

$$z = \frac{1}{4m \sin^2\theta}\left[\frac{du}{d\theta} + 2u \cot\theta\right]. \tag{24}$$

When $\theta = \pi/2$,*

$$z = \frac{1}{4m}\left[\frac{du}{d\theta}\right]_{\theta=\pi/2}.$$

* θ = co-latitude or polar distance.

$\therefore$ at the equator it is necessary that

$$\frac{du}{d\theta} = 0\,. \tag{25}$$

But

$$\frac{du}{d\theta} = \sqrt{1 - x^2}\,\frac{du}{dx}\,.$$

$\therefore$ it is necessary that,

$$\mathop{\pounds}_{x=1} \left\{ \sqrt{1 - x^2}\,\frac{du}{dx} \right\} = 0\,. \tag{26}$$

15. *Proof that $B = 0$.* In order to prove that the condition (26) requires that B shall be zero, the function will be considered in the neighborhood of $x = 1$.

Assume

$$x = 1 + y\,; \tag{27}$$

then

$$\frac{du}{dx} = \frac{du}{dy}\,, \quad \frac{d^2u}{dx^2} = \frac{d^2u}{dy^2}\,;$$

and equation (7) takes the form

$$\begin{aligned}(2y + 5y^2 + 4y^3 + y^4)\frac{d^2u}{dy^2} &+ (1 + y)\frac{du}{dy} \\ &+ u\{(6 - \beta) - 4(1 + \beta)y - 2(1 + 3\beta)y^2 - 4\beta y^3 - \beta y^4\} = 0\,.\end{aligned} \tag{28}$$

Theorems I and II apply to (28) also. Then assume

$$u = \sum_0^\infty{}_r\, A_r y^{m+rs}\,,$$

s being a positive integer. The equation for the determination of m is

$$2m^2 - m = 0\,; \tag{29}$$

whence $m = 0$ or $+\frac{1}{2}$. Also $s = 1$. The two integrals are expressible in series form. Moreover the relations connecting the A's are, for both values of m, such that each coefficient is given as a multiple of the first one. The two independent integrals of (28) may then be written

$$\left.\begin{aligned} y_1 &= 1 + \sum_1^\infty{}_r\, \alpha_r y^r\,, \\ y_2 &= y^{\frac{1}{2}} + \sum_1^\infty{}_r\, \beta_r y^{r+\frac{1}{2}}\,, \end{aligned}\right\} \tag{30}$$

the α's and β's being known.

The complete integral of (28), then, is

$$u = c_1 y_1 + c_2 y_2 . \tag{31}$$

If now the substitution (27) be made in (1), an equation results which differs from (28) only by the expression $- E_i^2\beta\,(1 + y)^6$ in the right hand member. Of this equation, the complementary function is given by (31). To complete its integration, it is necessary to find a particular integral. The character of y_1 and of the absolute term $- E_i^2\beta\,(1 + y)^6$ make it clear that a particular integral can be obtained in the form of a series of positive entire powers. Let this integral be denoted by Y. It is not of importance that its coefficients be calculated. Then, in the neighborhood of $x = 1$, or of $y = 0$, the integral of (28) can be expressed in the form

$$u = c_1 y_1 + c_2 y_2 + Y .$$

Then

$$\frac{du}{dx} = \frac{du}{dy} = c_1 \frac{dy_1}{dy} + c_2 \frac{dy_2}{dy} + \frac{dY}{dy} .$$

Also

$$\sqrt{1 - x^2} = \sqrt{-(y^2 + 2y)} = i y^{\frac{1}{2}} \sqrt{2 + y} ,$$

where $i = \sqrt{-1}$.

$$\therefore \quad \frac{du}{d\theta} = \frac{du}{dx} \sqrt{1 - x^2} = c_1 i y^{\frac{1}{2}} (2 + y)^{\frac{1}{2}} \frac{dy_1}{dy} + c_2 i y^{\frac{1}{2}} (2 + y)^{\frac{1}{2}} \frac{dy_2}{dy}$$
$$+ i y^{\frac{1}{2}} (2 + y)^{\frac{1}{2}} \frac{dY}{dy} . \tag{32}$$

When $y = 0$, the first and third terms on the right vanish, and the middle term becomes

$$[+ \tfrac{1}{2}\, y^{-\frac{1}{2}}\, c_2 i y^{\frac{1}{2}} (2 + y)^{\frac{1}{2}}]_{y=0}, \text{ or } c_2 i / \sqrt{2} .$$

$$\therefore \quad (du/d\theta)_{\theta = \pi/2} = c_2 i / \sqrt{2} . \tag{33}$$

In order, then, that at the equator $du/d\theta = 0$, the function must be such that $c_2 = 0$. But if $c_2 = 0$, the point $x = 1$ is an ordinary point of the function, while if $c_2 \gtrless 0$ the point $x = 1$ is a branch point of the function. If the point $x = 1$ is an ordinary point of the function, so also is $x = -1$.

In this case the function has no critical point in the finite part of the plane, and, if expressible in the neighborhood of the origin $x = 0$ by means of a Taylor's series, that series will converge for all finite values of the variable x. If the point $x = 1$ is a branch point of the function, and if the function is expressible in the neighborhood of the origin $x = 0$ by means of a Taylor's series, that series will have a circle of convergence of unit radius. It follows,

then, when $B = 0$ and $u = \mathfrak{L}$, that $c_2 = 0$ and $(du/d\theta)_{\theta=\pi/2} = 0$, and when $B \gtrless 0$, that $c_2 \gtrless 0$ and $(du/d\theta)_{\theta=\pi/2} \gtrless 0$.

Thus it is seen that the condition stated by Lord Kelvin requires that $B = 0$ and $u = \mathfrak{L}$. Consequently the series as written by Laplace is the complete solution for the case of an earth completely covered to a constant depth by water.

NOTE.—Equation (33) gives an imaginary value for $(du/d\theta)_{\theta=\pi/2}$ when the arbitrary constant c_2 is taken real. It is evident, however, that, when real values of the function between $x = 0$ and $x = 1$ are desired, c_2 must be taken purely imaginary; for it was assumed that

$$x = 1 + y\,; \tag{27}$$

so that when x is less than unity y is negative, and $y^{\frac{1}{2}}$ is a pure imaginary. The $c_2 y^{\frac{1}{2}}$ will be real when c_2 is purely imaginary, and it follows that then y_2 is also real.

VI.

DARWIN'S PRESENTATION OF LORD KELVIN'S PROOF THAT B MUST BE ZERO WHEN $(du/d\theta)_{\theta=\pi/2} = 0$.

16. *Darwin's argument.* The function $u = Bu_1 + \mathfrak{L}$ may be regarded as a single series of even positive integral powers commencing with the fourth and having the coefficient of x^4 arbitrary. It has already been seen what relations connect the coefficients and define them in terms of the coefficient of x^4 (A_2 say). It is known, too, that when

$$\mathop{\mathfrak{L}}_{n=\infty} \frac{A_{n+1}}{A_n} = 0$$

then $A_2 = L$.

Suppose now that

$$\mathop{\mathfrak{L}}_{n=\infty} \frac{A_{n+1}}{A_n} \gtrless 0\,,$$

but $= a^{-1}$, a finite quantity. Then

$$\begin{aligned} \frac{A_{n+2}}{A_{n+1}} &= \frac{2n+3}{2n+6} - \frac{\beta}{2n\,(2n+6)}\,\frac{A_n}{A_{n+1}} \\ &= \frac{2n+3}{2n+6} - \frac{\beta}{2n\,(2n+6)}\,(a-h)\,, \end{aligned} \tag{34}$$

where h tends to zero when n becomes indefinitely great. Darwin's argument is along the following lines: When

$$\mathop{\pounds}_{n=\infty} \frac{A_{n+1}}{A_n} \gtrless 0$$

then, for large values of n,

$$\frac{A_{n+2}x^{2n+4}}{A_{n+1}x^{2n+2}} = \frac{2n+3}{2n+6}x^2 = \left[1 - \frac{3}{2(n+3)}\right]x^2 = \left[1 - \frac{3}{2n}\right]x^2$$

nearly.

But if $(1 - x^2)^{\frac{1}{2}}$ be expanded by the binomial theorem the ratio of the $(n+1)$th term to the nth term is $\left[1 - \frac{3}{2n}\right]x^2$. Consequently, in this case, it is possible to write

$$u = A_1 + B_1(1 - x^2)^{\frac{1}{2}},$$

where A_1 and B_1 are finite for all values of x. A similar argument being made in the case of the series for du/dx it follows that

$$du/dx = C + D(1 - x^2)^{-\frac{1}{2}},$$

where C and D are finite (and not zero) for all values of x.

But

$$du/d\theta = du/dx\,(1 - x^2)^{\frac{1}{2}} = C(1 - x^2)^{\frac{1}{2}} + D;$$

$$\therefore\quad (du/d\theta)_{\theta=\pi/2} = [C(1 - x^2)^{\frac{1}{2}} + D]_{x=1} = D \gtrless 0.$$

17. *Discussion of Darwin's proof.* These results agree with each other, and with what has been proven in another way; but this proof of the fact that B_1 and D are not zero nor infinite does not appear to be entirely satisfactory, and it is essential that this property of B_1 and D be made evident. The ratio A_{n+2}/A_{n+1} becomes $1 - \frac{3}{2}n^{-1}$ when the square and higher powers of n^{-1} are neglected. If after a certain value of n quantities of the order n^{-1} be neglected, the ratio A_{n+2}/A_n becomes unity. Then, following a line of argument similar to that given, it would appear that

$$u = A_2 + B_2(1 - x^2)^{-1} \tag{a}$$

and, by a similar course of reasoning, that

$$du/dx = C_2 + D_2(1 - x^2)^{-1}.$$

These results do not agree and are incorrect, but they show in what respect

the previous reasoning is weak. For, suppose that the binomial expansion of $(1 - x^2)^{\frac{1}{2}}$ be written

$$(1 - x^2)^{\frac{1}{2}} = \sum_0^\infty {}_r \gamma_{2r} x^{2r} .$$

Then, if the infinite series can be written in the form

$$u = A_1 + B_1 (1 - x^2)^{\frac{1}{2}} ,$$

where A_1 and B_1 are finite for all values of x, it follows that

$$B_1 = \mathop{\pounds}_{r=\infty} \frac{A_r}{\gamma_r} .$$

Now for every finite value of n (β being positive),

$$\frac{A_{r+1}}{A_r} < \frac{\gamma_{r+1}}{\gamma_r} . \tag{b}$$

Darwin has shown that

$$\mathop{\pounds}_{r=\infty} \frac{A_{r+1}}{A_r} = \mathop{\pounds}_{r=\infty} \frac{\gamma_{r+1}}{\gamma_r} ;$$

but it is necessary also to show that

$$\frac{\mathop{\pounds}\limits_{r=\infty} A_1 \cdot \dfrac{A_2}{A_1} \cdot \dfrac{A_3}{A_2} \cdots \dfrac{A_r}{A_{r-1}} \cdot \dfrac{A_{r+1}}{A_r}}{\mathop{\pounds}\limits_{r=\infty} \gamma_1 \cdot \dfrac{\gamma_2}{\gamma_1} \cdot \dfrac{\gamma_3}{\gamma_2} \cdots \dfrac{\gamma_r}{\gamma_{r-1}} \cdot \dfrac{\gamma_{r+1}}{\gamma_r}} , \tag{c}$$

is finite.

Both numerator and denominator are zero so that the value of the quotient requires investigation.

The discussion of (a) and (c) shows that in (a) A_2 and B_2 do not have finite non-vanishing values for all values of x.

VII.

18. *Cases to be treated.* It remains, then, to examine the other cases included in the solution obtained. Airy pointed out that in the solution of the form

$$u = Bu_1(x) + \mathfrak{P}(x) ,$$

B could be determined so that the solution would be suitable for the case of a sea forming a spherical cap and extending from the pole to an arbitrary parallel of latitude. Lord Kelvin pointed out that, if the general solution were

at hand, the two constants could be determined so as to obtain a solution suitable for a zonal sea lying between two parallels of latitude. The most interesting cases to be dealt with appear then to be the following :—

1. *Case of a sea covering the whole earth. This is the case already treated.*

2. *Case of a sea extending from the pole to a given parallel of latitude.*

3. *Case of a zonal sea bounded by two parallels of latitude on opposite sides of the equator and equally distant from it.*

4. *Case of a zonal sea bounded by any two parallels of latitude lying in one hemisphere.*

5. *Case of a canal lying along a parallel of latitude.*

Case 2.

19. *Polar sea.* If the sea extends only to a given parallel of latitude from the pole, it is necessary that the meridional component of the motion should vanish for the corresponding value of x, the sine of the polar distance of the boundary.

Then, as in Case 1, $A = 0$, and the condition just named gives for the boundary value of x

$$\frac{du}{d\theta} + 2u \cot \theta = 0 .$$

Let, then, θ_1 be the colatitude of the southern boundary, and $\sin \theta_1 = a_1$.

Since

$$\frac{du}{d\theta} = \frac{du}{dx} \cos \theta , \text{ and } \cos \theta_1 \gtrless 0 ,$$

$$\therefore \qquad \frac{du}{dx} + \frac{2u}{\sin \theta} = 0 , \text{ for } \theta = \theta_1 ;$$

$$\therefore \qquad Bu_1'(a_1) + \mho'(a_1) + \frac{2}{a_1} \{ Bu_1(a_1) + \mho(a_1) \} = 0 ;$$

$$\therefore \qquad B = - \frac{a_1 \mho'(a_1) + 2\mho(a_1)}{a_1 u_1'(a_1) + 2u_1(a_1)} . \tag{35}$$

This gives the expression for u at any point in the form

$$u = \mho(x) - \frac{a_1 \mho'(a_1) + 2\mho(a_1)}{a_1 u_1'(a_1) + 2u_1(a_1)} u_1(x) . \tag{36}$$

In particular, for the southern boundary,

$$u(a_1) = a_1 . \frac{\mho(a_1) u_1'(a_1) - \mho'(a_1) u_1(a_1)}{a u_1'(a_1) + 2u_1(a_1)} . \tag{37}$$

(36) and (37) give the amounts to be added to the tide deduced from the equilibrium theory.

The total tide at any point then is

$$U = u(x) + Ex^2\,, \tag{38}$$

and at the southern boundary

$$U = u(a_1) + Ea_1^2\,. \tag{39}$$

As has been stated, Ferrel's calculations were made for the series in which B was given by the relation

$$B + L = 0\,.$$

Then, from (35) it appears that the tides calculated by Ferrel would be those existing on a circumpolar sea bounded by a parallel of latitude $\frac{1}{2}\pi - \theta$, where $a = \sin\theta$ and satisfies the equation

$$L = \frac{a\Upsilon'(a) + 2\Upsilon(a)}{au_1'(a) + 2u_1(a)}\,. \tag{39a}$$

CASE 3.

20. *Sea extending equally on both sides of equator.* Suppose the sea to extend equally on both sides of the equator, the boundaries being parallels of latitude.

The condition that there shall be no motion of water along the meridian at any point of the northern boundary gives one relation connecting A and B; but it is clear that the corresponding condition for the southern boundary gives exactly the same relation; so that one of the constants appears to be arbitrary. The considerations which applied to Case 1 apply to this case. The symmetry of the motion requires that there be no meridional motion of the water at the equator. In this case, also, it is necessary that

$$(du/d\theta)_{\theta=\pi/2} = 0\,.$$

This gives a second condition by means of which the remaining arbitrary constant may be determined.

From

$$u = A\,\{\varphi + u_1 \log x\} + Bu_1 + \Upsilon \tag{22}$$

it follows that

$$\frac{du}{d\theta} = A\,(1 - x^2)^{\frac{1}{2}}\,(\varphi' + u_1' \log x + \frac{1}{x}\,u_1) + B(1 - x^2)^{\frac{1}{2}}\,u_1' + (1 - x^2)^{\frac{1}{2}}\,\Upsilon'.$$

Now

$$\underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} \psi' \right] = 0 \, ;$$

$$\underset{x=1}{\mathcal{L}} \left[\frac{(1-x^2)^{\frac{1}{2}}}{x} u_1 \right] = 0 \, ,$$

since u_1 converges for $x = 1$;*

$$\underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} u_1' \log x \right] = 0 \, ,$$

since $\log 1 = 0$ and $\underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} u_1' \right]$ is finite.

It is clear, then, that

$$A \underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} \varphi' \right] + B \underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} u_1 \right] = 0 \, .$$

It has been seen that

$\underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} u_1' \right]$ is a finite quantity, say b ;

$\underset{x=1}{\mathcal{L}} \left[(1-x^2)^{\frac{1}{2}} \varphi' \right]$ is a finite quantity, say a .

(In one particular case it is possible that a might be zero.) Then it follows that

$$Aa + Bb = 0 \, . \tag{40}$$

Returning now to the condition first stated, let θ_1 be the colatitude of the boundary, and let $\sin \theta_1 = a_1$. It is necessary that

$$\left[\frac{du}{dx} + \frac{2u}{x} \right]_{x=a_1} = 0 \, , \tag{41}$$

since $\cos \theta_1 \gtrless 0$. The resulting relation between A and B takes the form

$$A \left[\varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u_1'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1(a_1) \log a_1 \right]$$
$$+ B \left[u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \right] + \psi'(a_1) + \frac{2}{a_1} \psi(a_1) = 0 \, . \tag{42}$$

* See equation (b) Section VI. The expansion of $(1-x^2)^{\frac{1}{2}}$ converges for $x = 1$.

An equation for u is obtained by eliminating A and B from (22), (40), and (42). The total tide is

$$u + Ex^2 .$$

Case 4.

21. *Sea bounded by two parallels of latitude on the same side of equator.* Suppose the sea to be bounded on the north and south by parallels of latitude and to lie entirely within one hemisphere.

It is necessary that, at the northern and southern boundaries,

$$\frac{du}{dx} + \frac{2}{x} u = 0 .$$

Let the boundaries have colatitudes θ_1, θ_2 $(\theta_1 < \theta_2)$, and let $\sin \theta_1 = a_1$, $\sin\theta_2 = a_2$.

Then the equations for the determination of A and B are similar to (42) and are

$$\left.\begin{aligned} &A\left[\varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + u_1'(a_1)\log a_1 + \frac{1}{a_1}u_1(a_1) + \frac{2}{a_1}u_1(a_1)\log a_1\right] \\ &\qquad\qquad + B\left[u_1'(a_1) + \frac{2}{a_1}u_1(a_1)\right] + \mathfrak{Y}'(a_1) + \frac{2}{a_1}\mathfrak{Y}(a_1) = 0 , \\ &A\left[\varphi'(a_2) + \frac{2}{a_2}\varphi(a_2) + u_1'(a_2)\log a_2 + \frac{1}{a_2}u_1(a_2) + \frac{2}{a_2}u_1(a_2)\log a_2\right] \\ &\qquad\qquad + B\left[u_1'(a_2) + \frac{2}{a_2}u_1(a_2)\right] + \mathfrak{Y}'(a_2) + \frac{2}{a_2}\mathfrak{Y}(a_2) = 0 . \end{aligned}\right\} \quad (43)$$

As before,

$$A\left[\varphi(x) + u_1(x)\log x\right] + Bu_1(x) + \mathfrak{Y}(x) - u = 0 . \qquad (22)$$

Then, eliminating A and B, the equation for u is

$$\begin{vmatrix} \varphi(x) + u_1(x)\log x & , u_1(x) & , \mathfrak{Y}(x) \\ \varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + u_1'(a_1)\log a_1 + \frac{1}{a_1}u_1(a_1) + \frac{2}{a_1}u_1(a_1)\log a_1, & u_1'(a_1) + \frac{2}{a_1}u_1(a_1), & \mathfrak{Y}'(a_1) + \frac{2}{a_1}\mathfrak{Y}(a_1) \\ \varphi'(a_2) + \frac{2}{a_2}\varphi(a_2) + u_1'(a_2)\log a_2 + \frac{1}{a_2}u_1(a_2) + \frac{2}{a_2}u_1(a_2)\log a_2, & u_1'(a_2) + \frac{2}{a_2}u_1(a_2), & \mathfrak{Y}'(a_2) + \frac{2}{a_2}\mathfrak{Y}(a_2) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + u_1'(a_1)\log a_1 + \frac{1}{a_1}u_1(a_1) + \frac{2}{a_1}u_1(a_1)\log a_1, & u_1'(a_1) + \frac{2}{a_1}u_1(a_1) \\ \varphi'(a_2) + \frac{2}{a_2}\varphi(a_2) + u_1'(a_2)\log a_2 + \frac{1}{a_2}u_1(a_2) + \frac{2}{a_2}u_1(a_2)\log a_2, & u_1'(a_2) + \frac{2}{a_2}u_1(a_2) \end{vmatrix} u . \qquad (44)$$

(44) gives the value of u. The complete tidal expression is

$$u + Ex^2.$$

The value of u at the boundary whose colatitude is θ_1 is given by

$$\begin{vmatrix} \varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + u'(a_1)\log a_1 + \frac{1}{a_1}u_1(a_1) + \frac{2}{a_1}u_1(a_1)\log a_1, & u_1'(a_1) + \frac{2}{a_1}u_1(a_1) \\ \varphi'(a_2) + \frac{2}{a_2}\varphi(a_2) + u'(a_2)\log a_2 + \frac{1}{a_2}u_1(a_2) + \frac{2}{a_2}u_1(a_2)\log a_2, & u_1'(a_2) + \frac{2}{a_2}u_1(a_2) \end{vmatrix} u(a_1)$$

$$= \begin{vmatrix} \varphi(a_1) & , u_1(a_1) & , \wp(a_1) \\ \varphi'(a_1) + \frac{1}{a_1}u_1(a_1) & , u_1'(a_1) & , \wp'(a_1) \\ \varphi'(a_2) + \frac{2}{a_2}\varphi(a_2) + u_1'(a_2)\log\frac{a_2}{a_1} + \frac{1}{a_2}u_1(a_2) + \frac{2}{a_2}u_1(a_2)\log\frac{a_2}{a_1}, & u_1'(a_2) + \frac{2}{a_2}u_1(a_2), & \wp'(a_2) + \frac{2}{a_2}\wp(a_2) \end{vmatrix} . \quad (44A)$$

A similar equation may be obtained for the evaluation of $u(a_2)$.

Case 5.

22. *Canal of width 2d lying along a parallel of latitude.* Suppose the zonal sea to narrow down to a canal of width $2d$ having as its northern boundary the parallel of latitude $\frac{1}{2}\pi - \theta$.

All three of the functions in the expression (22) are expressible in the neighborhood of $\theta = \theta_1$ in series form. Let them be expressed in this manner and let d be taken so small that powers of it higher than the first may for purposes of calculation be neglected.

The second of the equations (43) then becomes, after simplification by means of the first one,

$$\begin{aligned} A \Big[\varphi''(a_1) + \frac{2}{a_1}\varphi'(a_1) - \frac{2}{a_1^2}\varphi(a_1) + \frac{2}{a_1}u_1'(a_1) + \frac{1}{a_1^2}u_1(a_1) & \\ + [u_1''(a_1) + \frac{2}{a_1}u_1'(a_1) - \frac{2}{a_1^2}u_1(a_1)]\log a_1 \Big] & \\ + B\Big[u_1''(a_1) + \frac{2}{a_1}u_1'(a_1) - \frac{2}{a_1^2}u_1(a_1)\Big] + \wp''(a_1) + \frac{2}{a_1}\wp'(a_1) - \frac{2}{a_1^2}\wp(a_1) & \\ + d(lA + mB + n)\sqrt{1 - a_1^2} = 0, & \quad (45) \end{aligned}$$

where l, m, n may be easily found in terms of a_1.

Then the equation for the determination of u becomes

$$\begin{vmatrix} \varphi(x) + u_1(x)\log x & u_1(x) & \chi(x) - u \\ \varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + u_1'(a_1)\log a_1 + \frac{1}{a_1}u_1(a_1) + \frac{2}{a_1}u_1(a_1)\log a_1, & u_1'(a_1) + \frac{2}{a_1}u_1(a_1) & \chi'(a_1) + \frac{2}{a_1}\chi(a_1) \\ \left\{\begin{matrix} \varphi''(a_1) + \frac{2}{a_1}\varphi'(a_1) - \frac{2}{a_1^2}\varphi(a_1) + \frac{2}{a_1}u_1'(a_1) + \frac{1}{a_1^2}u_1(a_1) \\ + \log a_1[u_1''(a_1) + \frac{2}{a_1}u_1'(a_1) - \frac{2}{a_1^2}u_1(a_1)] + dl\sqrt{1-a_1^2} \end{matrix}\right\}, & \left\{\begin{matrix} u_1''(a_1) + \frac{2}{a_1}u_1'(a_1) - \frac{2}{a_1^2}u_1(a_1) \\ + dm\sqrt{1-a_1^2} \end{matrix}\right\}, & \left\{\begin{matrix} \chi''(a_1) + \frac{2}{a_1}\chi'(a_1) \\ - \frac{2}{a_1^2}\chi(a_1) + dn\sqrt{1-a_1^2} \end{matrix}\right\} \end{vmatrix} = 0\,. \quad (46)$$

This reduces to the form

$$\begin{vmatrix} \varphi(x) + u_1(x)\log\frac{x}{a_1} & u_1(x) & \chi(x) - u \\ \varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + \frac{1}{a_1}u_1(a_1) & u_1'(a_1) + \frac{2}{a_1}u_1(a_1) & \chi'(a_1) + \frac{2}{a_1}\chi(a_1) \\ \left\{\begin{matrix} \varphi''(a_1) + \frac{2}{a_1}\varphi'(a_1) - \frac{2}{a_1^2}\varphi(a_1) + \frac{2}{a_1}u_1'(a_1) \\ + \frac{1}{a_1^2}u_1(a_1) + d(l - m\log a_1)\sqrt{1-a_1^2} \end{matrix}\right\}, & \left\{\begin{matrix} u_1''(a_1) + \frac{2}{a_1}u_1'(a_1) - \frac{2}{a_1^2}u_1(a_1) \\ + dm\sqrt{1-a_1^2} \end{matrix}\right\}, & \left\{\begin{matrix} \chi''(a_1) + \frac{2}{a_1}\chi'(a_1) \\ - \frac{2}{a_1^2}\chi(a_1) + dn\sqrt{1-a_1^2} \end{matrix}\right\} \end{vmatrix} = 0\,,$$

and again to

$$\begin{vmatrix} \varphi(x) + u_1(x)\log\frac{x}{a_1} & u_1(x) & \chi(x) - u \\ \varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + \frac{1}{a_1}u_1(a_1) & u_1'(a_1) + \frac{2}{a_1}u_1(a_1) & \chi'(a_1) + \frac{2}{a_1}\chi(a_1) \\ \left\{\begin{matrix} \varphi''(a_1) + \frac{3}{a_1}\varphi'(a_1) + \frac{2}{a_1}u_1'(a_1) \\ + \frac{2}{a_1^2}u_1(a_1) + d(l - m\log a_1)\sqrt{1-a_1^2} \end{matrix}\right\}, & \left\{\begin{matrix} u_1''(a_1) + \frac{3}{a_1}u_1'(a_1) \\ + dm\sqrt{1-a_1^2} \end{matrix}\right\}, & \left\{\begin{matrix} \chi''(a_1) + \frac{3}{a_1}\chi'(a_1) \\ + dn\sqrt{1-a_1^2} \end{matrix}\right\} \end{vmatrix} = 0\,. \quad (47)$$

23. *Tide at point distant δ from boundary of canal.* **For any point** within the canal distant δ from the northern boundary $x = a_1 + \delta\sqrt{1 - a_1^2}$. This substitution being made and the product $d\delta$ being neglected the result is

$$
\begin{vmatrix}
\varphi(a_1) & , u_1(a_1) & , \wp(a_1) \\
\varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + \frac{1}{a_1}u_1(a_1) & , u_1'(a_1) + \frac{2}{a_1}u_1(a_1), & \wp'(a_1) + \frac{2}{a_1}\wp(a_1) \\
\varphi''(a_1) + \frac{3}{a_1}\varphi'(a_1) + \frac{2}{a_1}u_1'(a_1) + \frac{2}{a_1^2}u_1(a_1), & u_1''(a_1) + \frac{3}{a_1}u_1'(a_1), & \wp''(a_1) + \frac{3}{a_1}\wp'(a_1)
\end{vmatrix}
$$

$$
+ d\sqrt{1 - a_1^2}
\begin{vmatrix}
\varphi(a_1) & , u_1(a_1) & , \wp(a_1) \\
\varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + \frac{1}{a_1}u_1(a_1), & u_1'(a_1) + \frac{2}{a_1}u_1(a_1), & \wp'(a_1) + \frac{2}{a_1}\wp(a_1) \\
l - m\log a_1 & , m & , n
\end{vmatrix}
$$

$$
+ \delta\sqrt{1 - a_1^2}
\begin{vmatrix}
\varphi'(a_1) + \frac{1}{a_1}u_1(a_1) & , u_1'(a_1) & , \wp'(a_1) \\
\varphi'(a_1) + \frac{1}{a_1}u_1(a_1) + \frac{2}{a_1}\varphi(a_1) & , u_1'(a_1) + \frac{2}{a_1}u_1(a_1), & \wp'(a_1) + \frac{2}{a_1}\wp(a_1) \\
\varphi''(a_1) + \frac{3}{a_1}\varphi'(a_1) + \frac{2}{a_1}u_1'(a_1) + \frac{2}{a_1^2}u_1(a_1), & u_1''(a_1) + \frac{3}{a_1}u_1'(a_1), & \wp''(a_1) + \frac{3}{a_1}\wp'(a_1)
\end{vmatrix}
$$

$$
= u
\begin{vmatrix}
\varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + \frac{1}{a_1}u_1(a_1) & , u_1'(a_1) + \frac{2}{a_1}u_1(a_1) \\
\varphi''(a_1) + \frac{3}{a_1}\varphi'(a_1) + \frac{2}{a_1}u_1'(a_1) + \frac{2}{a_1^2}u_1(a_1), & u_1''(a_1) + \frac{3}{a_1}u_1'(a_1)
\end{vmatrix}
$$

$$
+ u\sqrt{1 - a_1^2}\, d
\begin{vmatrix}
\varphi'(a_1) + \frac{2}{a_1}\varphi(a_1) + \frac{1}{a_1}u_1(a_1), & u_1'(a_1) + \frac{2}{a_1}u_1(a_1) \\
l - m\log a_1 & , m
\end{vmatrix}. \tag{48}
$$

It is easy to show that

$$
\begin{aligned}
l - m\log a_1 &= \varphi'''(a_1) + \frac{2}{a_1}\varphi''(a_1) - \frac{4}{a_1^2}\varphi'(a_1) + \frac{4}{a_1^3}\varphi(a_1) \\
&\quad + \frac{3}{a_1}u_1''(a_1) + \frac{1}{a_1^2}u_1'(a_1) - \frac{4}{a_1^3}u_1(a_1), \\
m &= u_1'''(a_1) + \frac{2}{a_1}u_1''(a_1) - \frac{4}{a_1^2}u'(a_1) + \frac{4}{a_1^3}u_1(a_1), \\
n &= \wp'''(a_1) + \frac{2}{a_1}\wp''(a_1) - \frac{4}{a_1^2}\wp'(a_1) + \frac{4}{a_1^3}\wp(a_1).
\end{aligned}
$$

Equation (48) can then be put in the form

$$u \begin{vmatrix} \varphi'(a_1) + \dfrac{2}{a_1}\varphi(a_1) + \dfrac{1}{a_1}u_1(a_1) & , u_1'(a_1) + \dfrac{2}{a_1}u_1(a_1) \\ \varphi''(a_1) + \dfrac{3}{a_1}\varphi'(a_1) + \dfrac{2}{a_1}u_1'(a_1) + \dfrac{2}{a_1^2}u_1(a_1), & u_1''(a_1) + \dfrac{3}{a_1}u_1'(a_1) \end{vmatrix}$$

$$+ du\sqrt{1-a_1^2} \begin{vmatrix} \varphi'(a_1) + \dfrac{2}{a_1}\varphi(a_1) + \dfrac{1}{a_1}u_1(a_1) & , u_1'(a_1) + \dfrac{2}{a_1}u_1(a_1) \\ \left\{ \begin{matrix} \varphi'''(a_1) + \dfrac{2}{a_1}\varphi''(a_1) + \dfrac{12}{a_1^3}\varphi(a_1) \\ + \dfrac{3}{a_1}u_1''(a_1) + \dfrac{1}{a_1^2}u_1'(a_1) \end{matrix} \right\}, & \left\{ \begin{matrix} u_1'''(a_1) + \dfrac{2}{a_1}u_1''(a_1) \\ + \dfrac{12}{a_1^3}u_1(a_1) \end{matrix} \right\} \end{vmatrix}$$

$$= \begin{vmatrix} \varphi(a_1) & , u_1(a_1) & , \Upsilon(a_1) \\ \varphi'(a_1) + \dfrac{1}{a_1}u_1(a_1) & , u_1'(a_1), & \Upsilon'(a_1) \\ \varphi''(a_1) + \dfrac{2}{a_1}u_1'(a_1) - \dfrac{1}{a_1^2}u_1(a_1), & u_1''(a_1), & \Upsilon''(a_1) \end{vmatrix} \left[1 + \frac{2(d-\delta)\sqrt{1-a_1^2}}{a_1}\right]$$

$$+ d\sqrt{1-a_1^2} \begin{vmatrix} \varphi(a_1) & , u_1(a_1) & , \Upsilon(a_1) \\ \varphi'(a_1) + \dfrac{1}{a_1}u_1(a_1) & , u_1'(a_1) & , \Upsilon'(a_1) \\ \varphi'''(a_1) + \dfrac{3}{a_1}u_1''(a_1) - \dfrac{3}{a_1^2}u_1'(a_1) + \dfrac{2}{a_1^3}u_1(a_1), & u_1'''(a_1), & \Upsilon'''(a_1) \end{vmatrix}. \quad (49)$$

The value of u at the middle of the canal is obtained by putting $\delta = d$, which simplifies (49) somewhat.

24. *Canal of negligible width.* If now the canal be taken so narrow that the width may be neglected, (49) is still further simplified. The results should coincide with those obtained by considering the motion as in two dimensions. In this case the equation for u is

$$u \begin{vmatrix} \varphi'(a_1) + \dfrac{2}{a_1}\varphi(a_1) + \dfrac{1}{a_1}u_1(a_1) & , u_1'(a_1) + \dfrac{2}{a_1}u_1(a_1) \\ \varphi''(a_1) + \dfrac{3}{a_1}\varphi'(a_1) + \dfrac{2}{a_1}u_1'(a_1) + \dfrac{2}{a_1^2}u_1(a_1), & u_1''(a_1) + \dfrac{3}{a_1}u_1'(a_1) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi(a_1) & , u_1(a_1) & , \Upsilon(a_1) \\ \varphi'(a_1) + \dfrac{1}{a_1}u_1(a_1) & , u_1'(a_1), & \Upsilon'(a_1) \\ \varphi''(a_1) + \dfrac{2}{a_1}u_1'(a_1) - \dfrac{1}{a_1^2}u_1(a_1), & u_1''(a_1), & \Upsilon''(a_1) \end{vmatrix}. \quad (50)$$

As before, the total tide is

$$U = u + Ea_1^2. \tag{51}$$

VIII.

Summary of Results.

25. *Summary of I–IV.* For convenience of reference, and in order to render the results available to any who do not desire to follow through the processes of obtaining them, it has been thought desirable that they should be restated in a separate section which, along with the historical sketch in Section II, would give a complete account of the state of the problem. Section III is devoted to a discussion and criticism of the analytical process by means of which Laplace obtained his value for the arbitrary constant in his solution. Objection is taken to the process employed for two reasons. In the first place, the reasoning used has not been shown to be, and does not appear to be strictly accurate. In this connection it may be said that in the examination of the apparent inaccuracies it has been thought sufficient to indicate the weaknesses of the method rather than go into a minute discussion of them. The modern advances in the theory of Differential Equations make it appear probable that matters of this character will be treated differently in future. In Section IV the complete solution of the equation is found, the expressions involved being infinite series, whose regions of convergence are large enough to make possible the treatment of all cases that can arise. The regions of convergence of the series together with certain important properties of the integrals can be predicted from the form of the equation. The integral found is more general than that previously deduced involving the two arbitrary constants. The series used by Laplace enters this integral as a part of it. In the derivation of the integral the method of Laplace as given in the Mécanique Céleste is made clear. In the closing paragraphs of the section certain properties of the integral important in the application of the physical conditions are deduced.

26. *Summary of V–VII.* Section V deals with the applications of the physical conditions to the determination of the arbitrary constants. One of the constants is immediately determined. The determination of the other involves greater difficulties. The analysis on which previous evaluations have been based is rejected and replaced by a determination of the value which appears to be entirely satisfactory. The condition which determines this constant is the condition stated by Lord Kelvin. The result of this section appears to be a complete justification of Laplace's series.

In Section VI the objections taken to the proof previously given in the determination of the second arbitrary constant are set forth.

The last Section contains the discussion of five important cases in the theory of tides. The arbitrary constants in the general integral of the differential equation are determined so that the integrals represent the tidal disturbance in these cases, and expressions are obtained for the tidal disturbance at any point whatever, and at certain particular points, such as points on the boundary. The last of the five cases treated is that of a canal lying along a parallel of latitude and would appear to furnish a means of checking the same case treated by Airy's Canal Theory of Tides.

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